Cosmological Hydrogen Recombination: The effect of high-n states

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Outline

- The history of high-n and recombination
- Our tools: RecSparse
- High-n and Recombination histories
- Quadrupole transitions
- Quadrupole transitions and Recombination histories
- Results: Recombination histories, effects on CMB
- Ongoing work

EQUILIBRIUM ASSUMPTIONS

- Radiative eq. between different n-states

$$\mathcal{N}_n = \mathcal{N}_2 e^{-(E_n - E_2)/T}$$

- Radiative/collisional eq. between different l

$$\mathcal{N}_{nl} = \mathcal{N}_n \frac{(2l+1)}{n^2}$$

- Matter in eq. with radiation due to Thompson scattering

$$T_m = T_\gamma \text{ since } \frac{\sigma_{\mathrm{T}} a T_\gamma^4 c}{m_e c^2} < H(T)$$

BREAKING THE NAIVE MODEL

- Radiation field is cool: ~~Boltzmann eq. of higher n~~
- Seager/Sasselov/Scott (2000) $n_{\max} = 300$ RecFAST!!!

- ~~Equilibrium~~ between *l states*
- Treated by Chluba et al. (2005) for $n_{\max} = 100$
- Radiation and matter field fall out of eq.

$$\dot{T_M} + 2HT_m = \frac{8x_e\sigma_{\mathrm{T}}aT_\gamma^4}{3m_ec\,(1+f_{\mathrm{He}}+x_e)}\,(T_M - T_\gamma)$$

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- ~~Equilibrium~~ between *l states*
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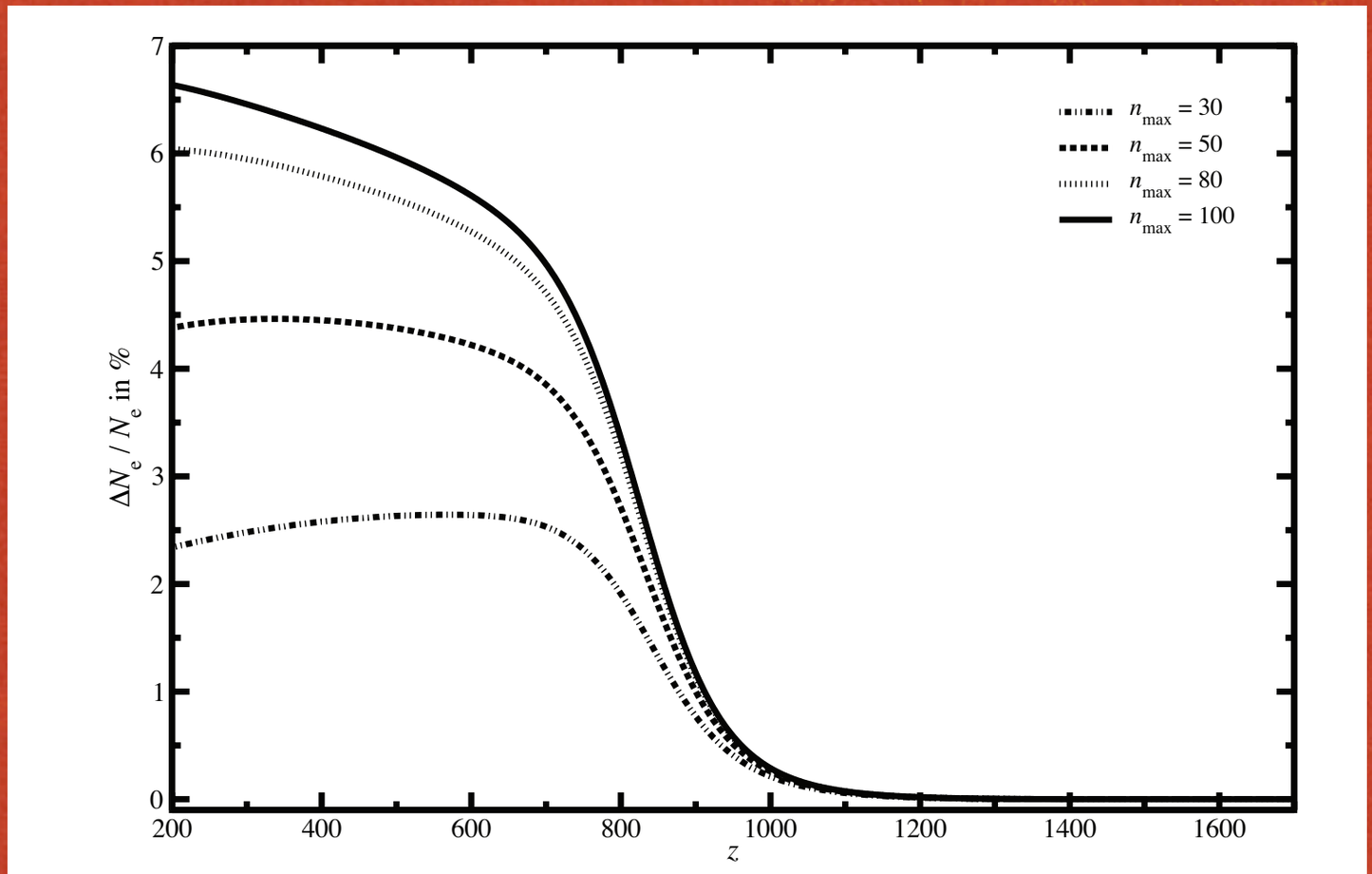
- Beyond this, testing convergence with $n_{\rm max}$ is hard!

$$t_{\rm compute} \sim \mathcal{O}\,(\text{weeks})$$

How to proceed if we want 0.1% accuracy in $x_e(z)$?

THE EFFECT OF RESOLVING 1- SUBSTATES



- Putting free-electrons in ‘bottlenecked’ l-substates slows down the decay to 1s: Recombination is slower; Chluba, Rubino-Martin, Sunyaev 2006

BREAKING THE NAIVE MODEL

- Radiation field is cool: ~~Boltzmann eq. of higher n~~
- Treated by Seager et al. (2000) $n_{\max} = 300$ RecFAST!!!

- ~~Eq.~~ between *l states*: dipole selection bottleneck: $\Delta l = \pm 1$
- Treated by Chluba et al. (2005) for $n_{\max} = 100$
- Beyond this, testing convergence with $n_{\max}$ is hard!

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THE MULTI-LEVEL ATOM (MLA)

- Bound-free rate equation

$$\dot{x}_{nl}^{bf} = \int dE_{\rm e} P_M(T_m, E_{\rm e}) n_H x_e x_p \left[1 + f(E_e - E_n)\right] \alpha_{nl}(E_{\rm e})$$
$$- \int dE_{\rm e}\ g(E_{\rm E} - E_n)\ x_{nl}\ f(E_{\rm e} - E_{nl}) \alpha_{nl}(E_{\rm E}) / g_{nl}$$

- Bound-bound rate equation

$$\dot{x}_{nl}^{bb} = \sum_{n',l'=l\pm1} (A_{nn'}^{ll'}(1 + f_{nn'})x_{n',l'} - \frac{g_{n'l'}}{g_{nl}} f_{nn'} x_{nl})\, P_{nn'}^{ll'}$$

THE MULTI-LEVEL ATOM (MLA)

- Bound-free rate equation

Ω_m, Ω_b, h

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- Phase-space density blueward of line
- Escape probability of γ in line

THE MULTI-LEVEL ATOM (MLA)

Stimulated emission/absorption

- Bound-free rate equation

$$\dot{x}^{bf}_{nl} = \int dE_{\rm e} P_M(T_m, E_{\rm e}) n_H x_e x_p \left[1 + f(E_e - E_n)\right] \alpha_{nl}(E_{\rm e})$$
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THE MULTI-LEVEL ATOM (MLA)

Spontaneous Emission

- Bound-free rate equation

$$\dot{x}_{nl}^{bf} = \int dE_{\rm e} P_M(T_m, E_{\rm e}) n_H x_e x_p \left[1 + f(E_e - E_n)\right] \alpha_{nl}(E_{\rm e})$$
$$- \int dE_{\rm e}\; g(E_{\rm E} - E_n)\; x_{nl}\; f(E_{\rm e} - E_{nl}) \alpha_{nl}(E_{\rm E}) / g_{nl}$$

- Bound-bound rate equation

$$\dot{x}_{nl}^{bb} = \sum_{n',l'=l\pm1} \left(A_{nn'}^{ll'} \left(1 + f_{nn'}\right) x_{n',l'} - \frac{g_{n'l'}}{g_{nl}} f_{nn'} x_{nl}\right) P_{nn'}^{ll'}$$

THE MULTI-LEVEL ATOM (MLA)

- Two photon transitions between n=1 and n=2 are included:

$$\dot{x}_{2s\to 1s,2\gamma} = -\dot{x}_{1s\to 2s,2\gamma} = \Lambda_{2s}(-x_{2s} + x_{1s}e^{-E_{2s\to 1s}/T_\gamma})$$

- Net recombination rate:

$$x_e \simeq 1 - x_{1s} \to \dot{x}_e \simeq -\dot{x}_{1s} = -\dot{x}_{1s\to 2s}$$
$$+\sum_{n,l>1s} A^{l0}_{n1} P^{l0}_{n1} \left\{ \frac{g_{nl}}{2} f^{+}_{n1} x_{1s} - (1 + f^{+}_{n1}) x_{nl} \right\}$$

Bound-bound Rate coefficients

- Bound-bound rates given by Fermi's golden rule and matrix element

$$\rho(n'l', nl) = \int_0^\infty u_{n'l'}(r) u_{nl}(r) r^3 dr = \mathcal{C} \times \left[F_{2,1}\left(-n+l+1, -n'+l, 2l, \frac{-4nn'}{(n-n')^2}\right) \right.$$
$$\left. - \left(\frac{n-n'}{n+n'}\right)^2 F_{2,1}\left(-n+l-1, -n'+l, 2l, \frac{-4nn'}{(n-n')^2}\right) \right]^2$$

- Power-series destabilizes at high-n, recursion relation used

- Rates are calculated, tabulated, and stored

BB Rate coefficients: verification

- WKB estimate of matrix elements $\rho(n'l', nl) = a_0 n^2 \int_{-\pi}^{\pi} d\tau e^{i\Omega\tau} (1 + cos\eta)$

Fourier transform of classical orbit!
Application of correspondence principle!

$$\Omega = \omega_n - \omega_{n'}$$
$$r = r_{\max} (1 + \cos\eta)/2$$
$$\tau = \eta + \sin\eta$$

$$\rho^{\text{dipole}}(n, l, n', l') = \frac{n_c^2}{s} \left\{ J_{s-1}(s\epsilon) - \frac{1 \mp \sqrt{1-\epsilon^2}}{\epsilon} J_s(s\epsilon) \right\}$$

$$\epsilon = \left(1 - \frac{l(l+1)}{n^2}\right)^{1/2}$$

$$s = n - n'$$

- Radial matrix elements checked against WKB (10%), published rates of Brocklehurst (1971), Green, Rush, and Chandler (1967) (agreement to their published 4 digits)

BOUND-FREE RATES

- Using continuum wave functions, bound-free rates are obtained (Burgess 1957)
- Bound-free matrix elements satisfy a convenient recursion relation:

$$\rho(n,l,\kappa,l') \equiv \sqrt{\frac{(n+l)!}{(n-l-1)!}\prod_{s=0}^{l'}(1+s^2\kappa^2)}(2n)^{l-n}G(n,l,\kappa,n) \qquad \kappa^2 \equiv K_e/\mathrm{Ryd}$$

$$G(n,l-2,\kappa,l-1) = \left[4n^2-4l^2+l(2l-1)(1+n^2\kappa^2)\right]G(n,l-1,\kappa,l) - 4n^2(n^2-l^2)\left[1+(l+1)^2\kappa^2\right]G(n,l,\kappa,l+1)$$

- For each n, dipole BF rates tabulated for 550 values of κ in 11 logarithmic bins from $n^2\kappa^2 = 10^{-10} \rightarrow 4.96\times10^8$
- Matrix elements compared with Burgess 1965 (5 digits) and with WKB approximation (5%): $\rho(n,l,\kappa,l\pm1) \simeq \frac{l^2}{\pi\sqrt{3}un^{3/2}}\left\{K_{2/3}(ul^3/3) \pm K_{1/3}(ul^3/3)\right\}$ $u = \frac{1}{2}(\kappa^2+\frac{1}{n^2})$
- At each temperature, thermal recombination/ionization rates obtained using 11-point Newton-Cotes formula, agreement with Burgess to 4 published digits

Radiation Field: Black Body+

- Escape probability treated in Sobolev approx.

$$P_{n,n'}^{l,l'} = \frac{1 - e^{-\tau_s}}{\tau_s}$$

$$\tau_s = \frac{c^3 n_{\rm H}}{8\pi H \nu_{nn'}^3} A_{nn'}^{ll'} \left[\frac{g_{n'}^{l'}}{g_n^l} x_n^l - x_{n'}^{l'} \right]$$

$$\mathcal{R}(\nu, \nu') = \phi(\nu)\phi(\nu') \qquad \frac{v_{\rm th}}{H(z)} \ll \lambda$$

- Excess line photons injected into radiation field

$$\left(\frac{8\pi\nu_{nn'}^3}{c^3 n_H} \right) \left(f_{nn'}^+ - f_{nn'}^- \right) = A_{nn'}^{ll'} P_{nn'}^{ll'} \left[x_n^l \left(1 + f_{nn'}^+ \right) - \frac{g_n^l}{g_{n'}^{l'}} x_{n'}^{l'} f_{nn'}^+ \right]$$

- Photons are conserved outside of line regions

$$f_{n1}^{+10} = f_{n+1,1}^{-10} \left[\frac{1 - (n+1)^{-2}}{1 - n^{-2}} (1+z) - 1 \right]$$

Radiation Field: Black Body+

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$$P^{l,l'}_{n,n'} = \frac{1 - e^{-\tau_s}}{\tau_s}$$

$$\tau_s = \frac{c^3 n_{\rm H}}{8\pi H \nu^3_{nn'}} A^{ll'}_{nn'} \left[\frac{g^{l'}_{n'}}{g^l_n} x^l_n - x^{l'}_{n'} \right]$$

$$\mathcal{R}(\nu, \nu') = \phi(\nu)\phi(\nu')$$

- Ali-Haimoud, Hirata, and Forbes are solving FP eqn. to obtain evolution of $f(\nu)$ more generally, including atomic recoil/diffusion, 2γ decay and full time-dependence of problem, coherent and incoherent scattering, overlap of higher-order Lyman lines

STEADY-STATE APPROXIMATION FOR EXCITED STATES

- Evolution equations may be re-written in matrix form

$$\frac{d\vec{x}}{dt} = \mathbf{R}\vec{x} + \vec{s}$$

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$$\vec{x} = \begin{pmatrix} \vec{x}_0 \\ \vec{x}_1 \\ \dots \\ \vec{x}_{n_{\max}-1} \end{pmatrix}$$

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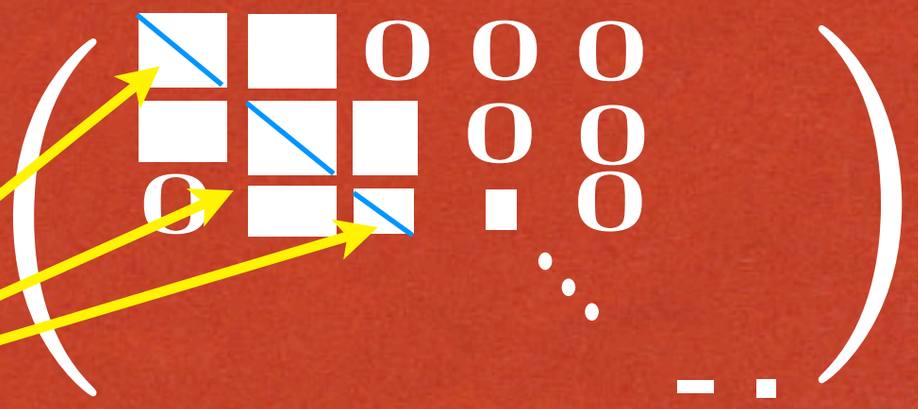

For state l, includes BB transitions out of l to all other l'', photo-ionization, 2γ transitions to ground state

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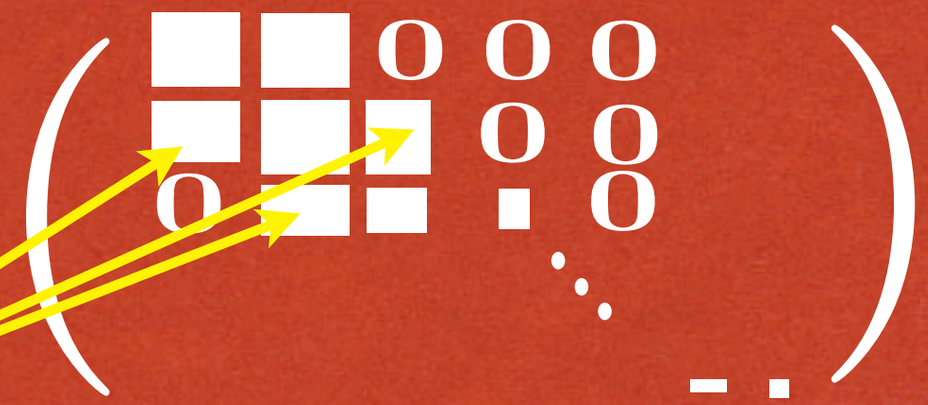


For state l, includes BB transitions into l from all other l'

STEADY-STATE APPROXIMATION FOR EXCITED STATES

- Evolution equations may be re-written in matrix form

$$\frac{d\vec{x}}{dt} = \mathbf{R}\vec{x} + \vec{s}$$

- Includes recombination to l, 1 and 2γ transitions from ground state

STEADY-STATE APPROXIMATION FOR EXCITED STATES

- Evolution equations may be re-written in matrix form

$$\frac{d\vec{x}}{dt} = \mathbf{R}\vec{x} + \vec{s}$$

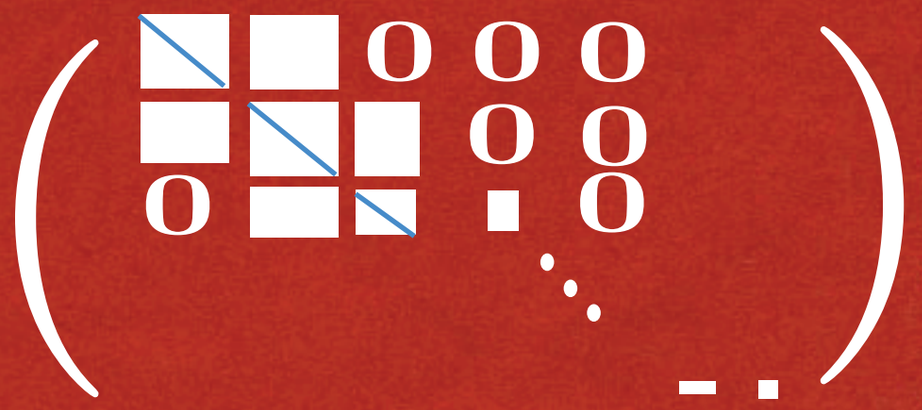
- For n>1, $t_{\rm rec}^{-1} \sim 10^{-12} s^{-1} \ll \mathbf{R}$, $\vec{s} \rightarrow \vec{x} \simeq \mathbf{R}^{-1}\vec{s}$

$\mathbf{R} \lesssim 1\ \mathrm{s}^{-1}$ (e.g. Lyman-α)

Rapid matrix inversion: Sparsity to the rescue

- Matrix is $\approx n_{max}^2 \times n_{max}^2$
- Brute force would require $n_{max}^6 \approx 1000$ s for $n_{max} = 200$ for a single time step
- Sparsity to the rescue $\Delta l = \pm 1$

$$M_{l,l-1}\vec{x}_{l-1} + M_{l,l}\vec{x}_l + M_{l,l+1}\vec{x}_{l+1} = \vec{s}_l$$



$$\vec{v}_l = \chi_1 \left[\vec{s}_l - \mathbf{M_{1,1+1}} \vec{v}_l + \Sigma_{l'=l-1}^{0} \sigma_{l,l'} \vec{s}_{l'} (-1)^{l'-l} \right]$$

$$\chi_l = \begin{cases} \mathbf{M}_{00}^{-1} & \text{if } l = 0 \\ (\mathbf{M}_{l+1,l+1} - \mathbf{M}_{l+1,l}\chi_l \mathbf{M}_{l,l+1})^{-1} & \text{if } l > 0 \end{cases}$$

$$\sigma_{l,l-1} = \mathbf{M}_{l,l-1}\chi_{l-1}$$
$$\sigma_{l,i} = \sigma_{l,i+1}\mathbf{M}_{i+1,i}\chi_i$$

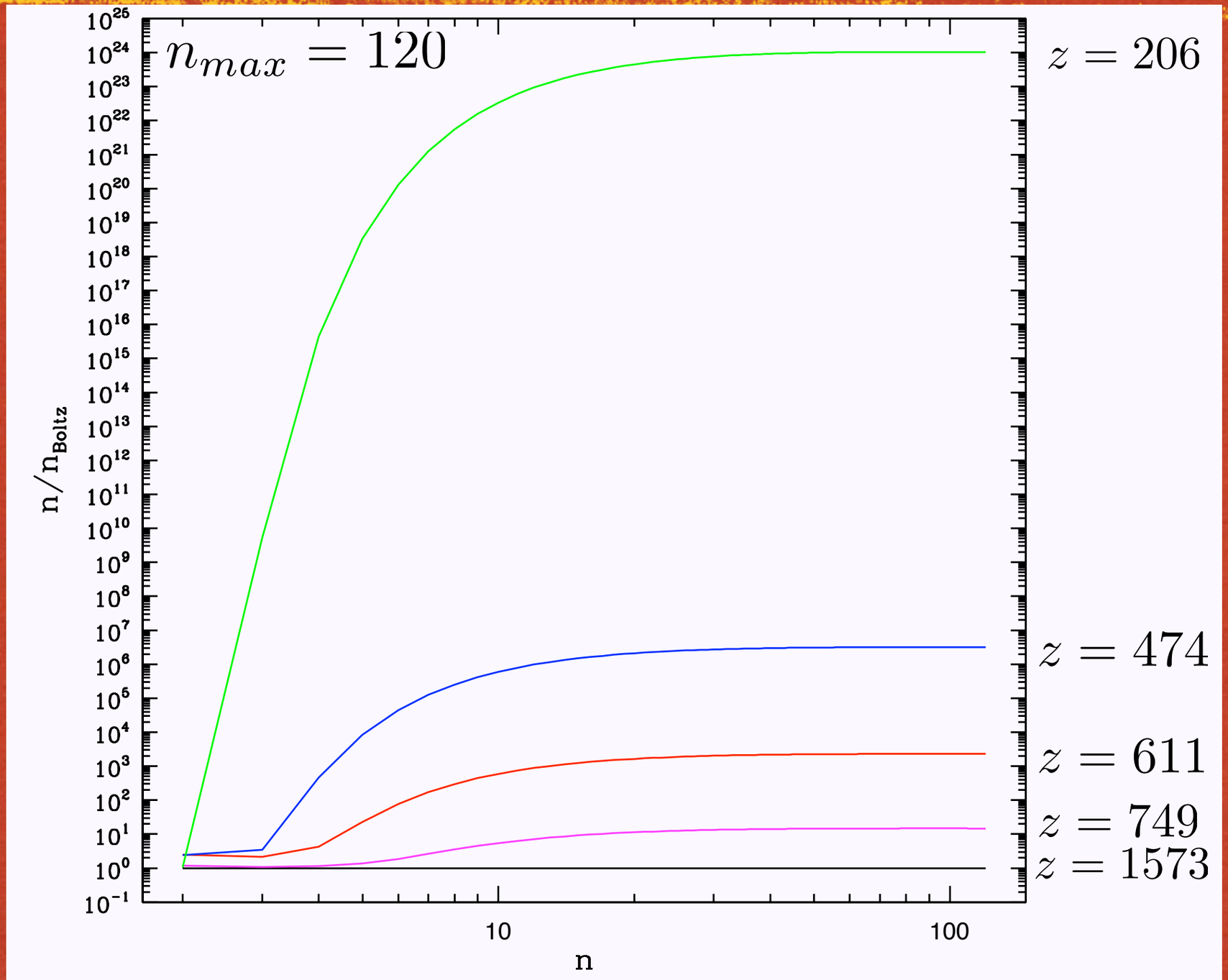
Some computational notes

- Ingredients incorporated into user-friendly code (RecSparse) which outputs x(z) for all times and atomic populations at several chosen slices.

- Collisions neglected for time being
- LAPACK libraries used for inversion of submatrices
- Simple rk4 ode solver used (inopportune for a stiff set of equations)
- Checked on MLA code of Hirata et al. with higher level two-photon transitions turned off and dense time grid (19548 steps in dlna going from z=1606 to z=700), agreement to several parts in 10^5, with and without feedback

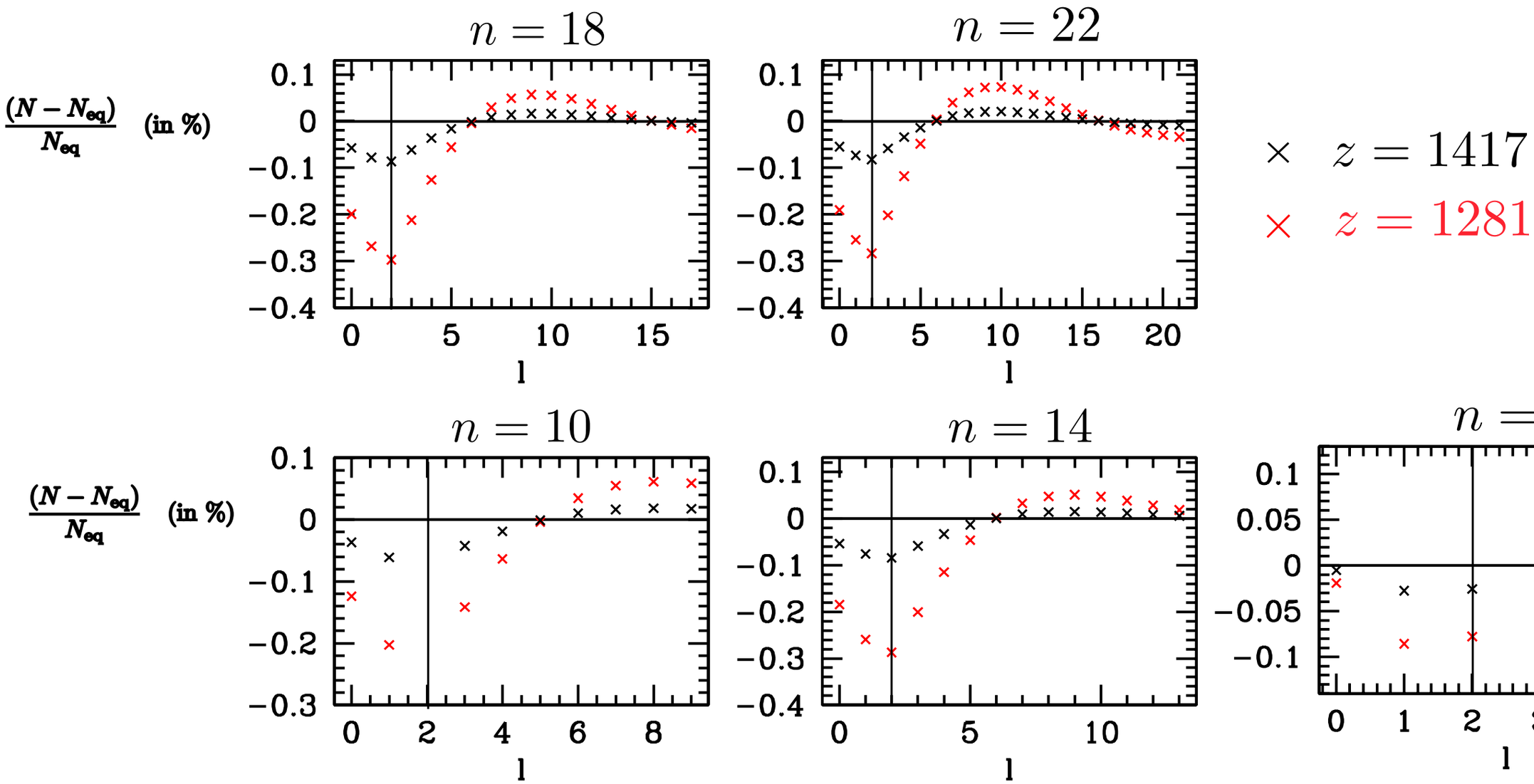
DEVIATIONS FROM BOLTZMANN EQ: HIGH-N

- $\alpha n \gtrsim A_{\rm bb,down}$.



DEVIATIONS FROM BOLTZMANN EQ: l-substates

RecSparse results $n_{\max} = 30$



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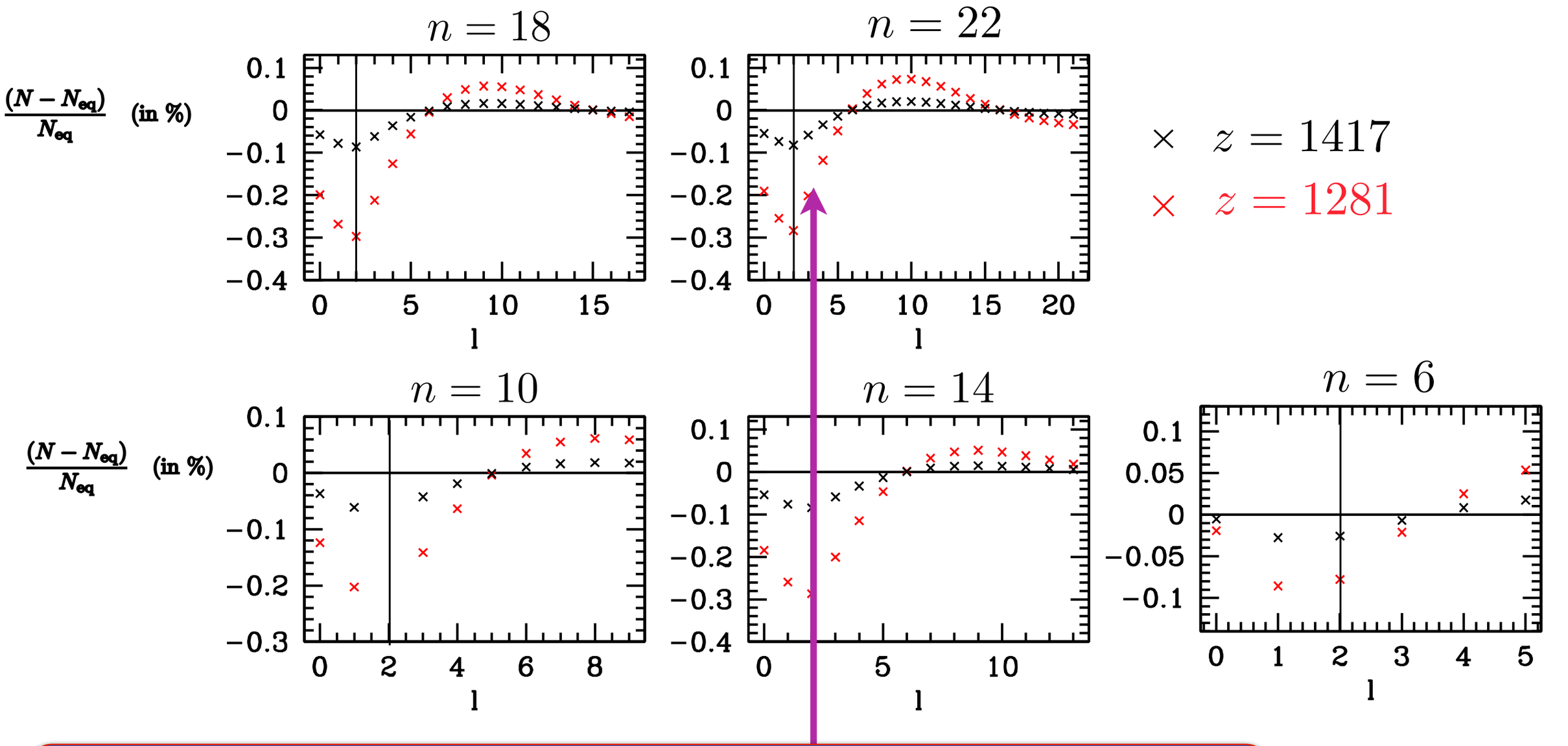

Lower l states can easily cascade down, and are relatively under-populated

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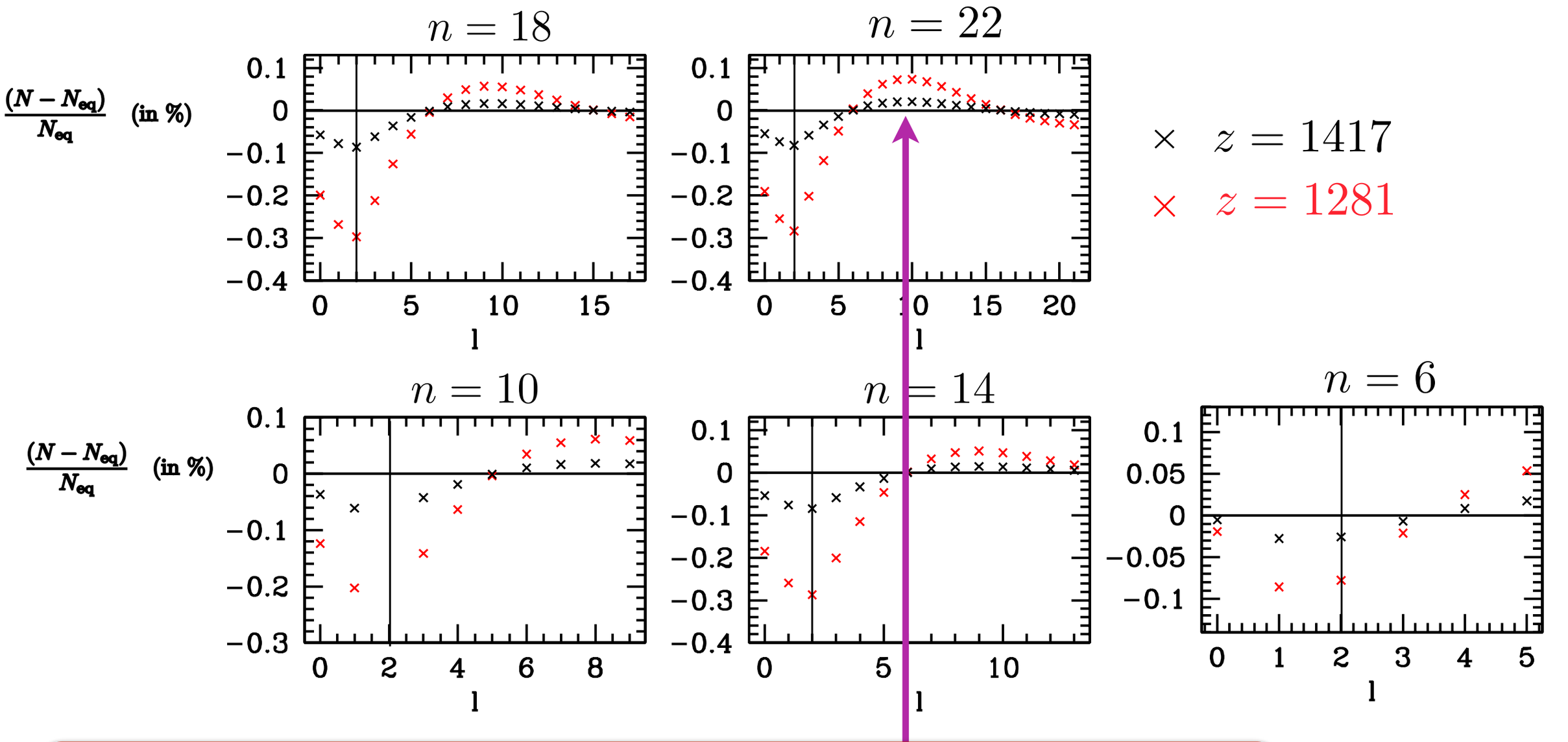

Higher l states can't easily cascade down, and are relatively over-populated

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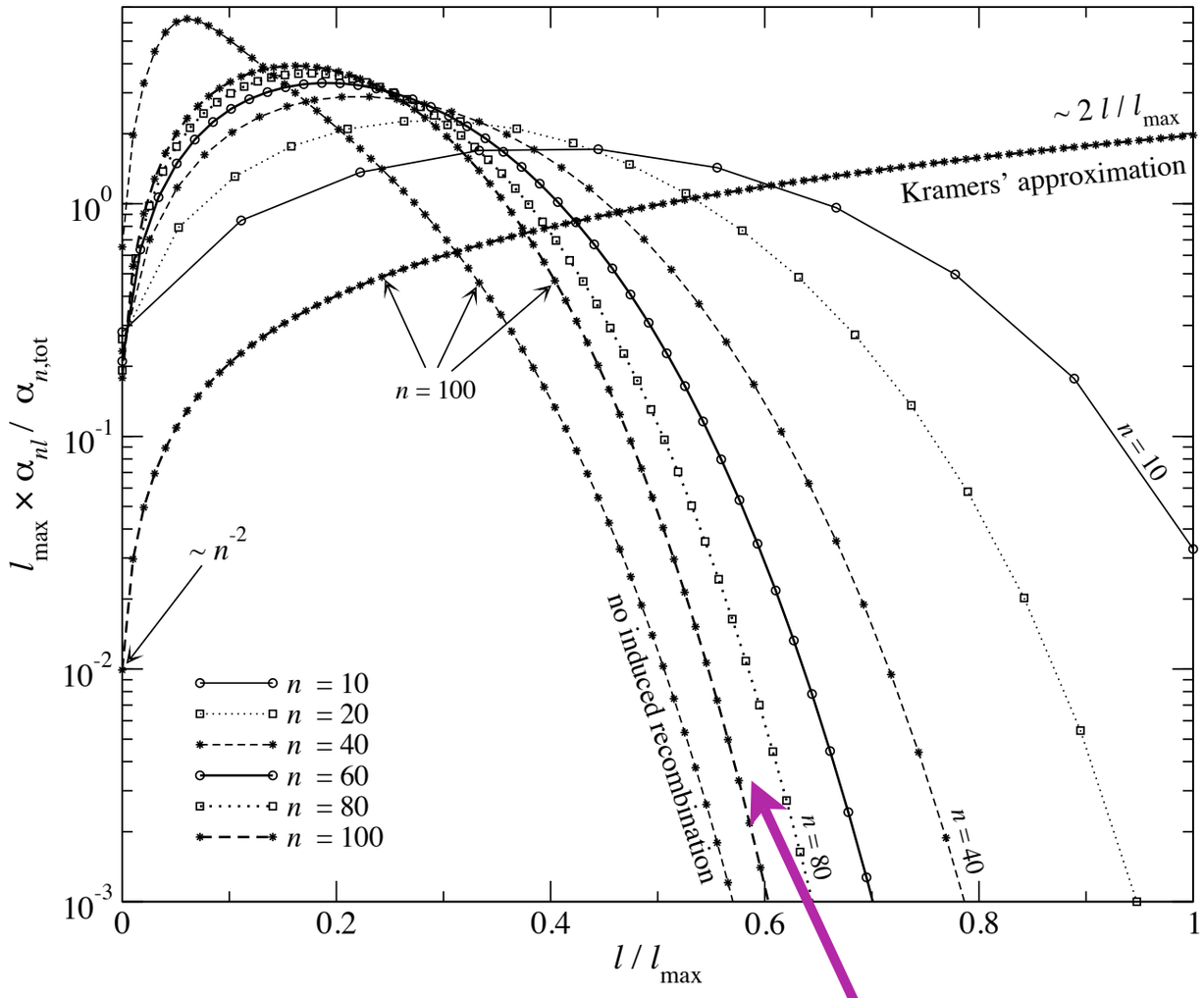
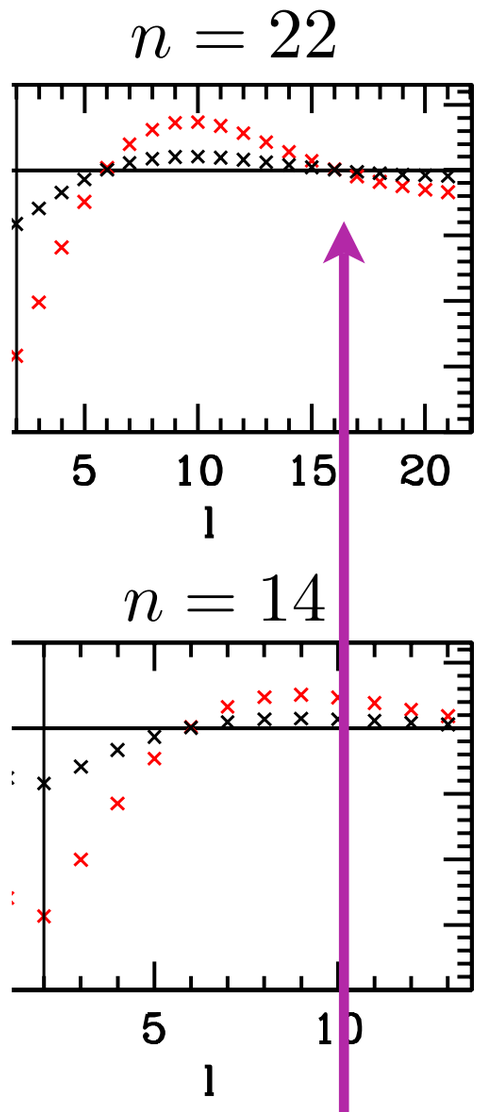
Chluba/Rubino-Martin/Sunyaev 2006





$\times \quad z = 1417$

$\times \quad z = 1281$

Highest l states recombine inefficiently, and are relatively under-populated

DEVIATIONS FROM BOLTZMANN EQ: l-substates

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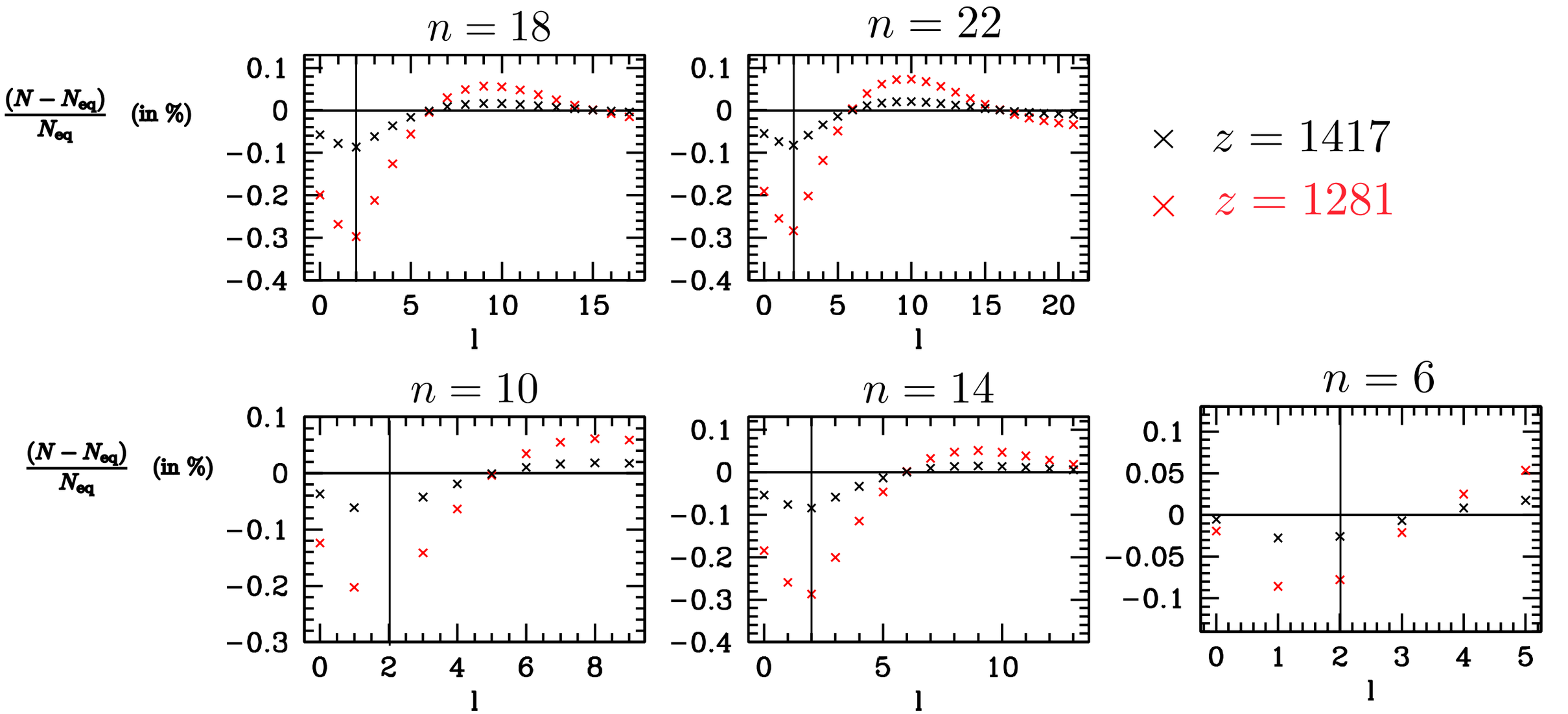

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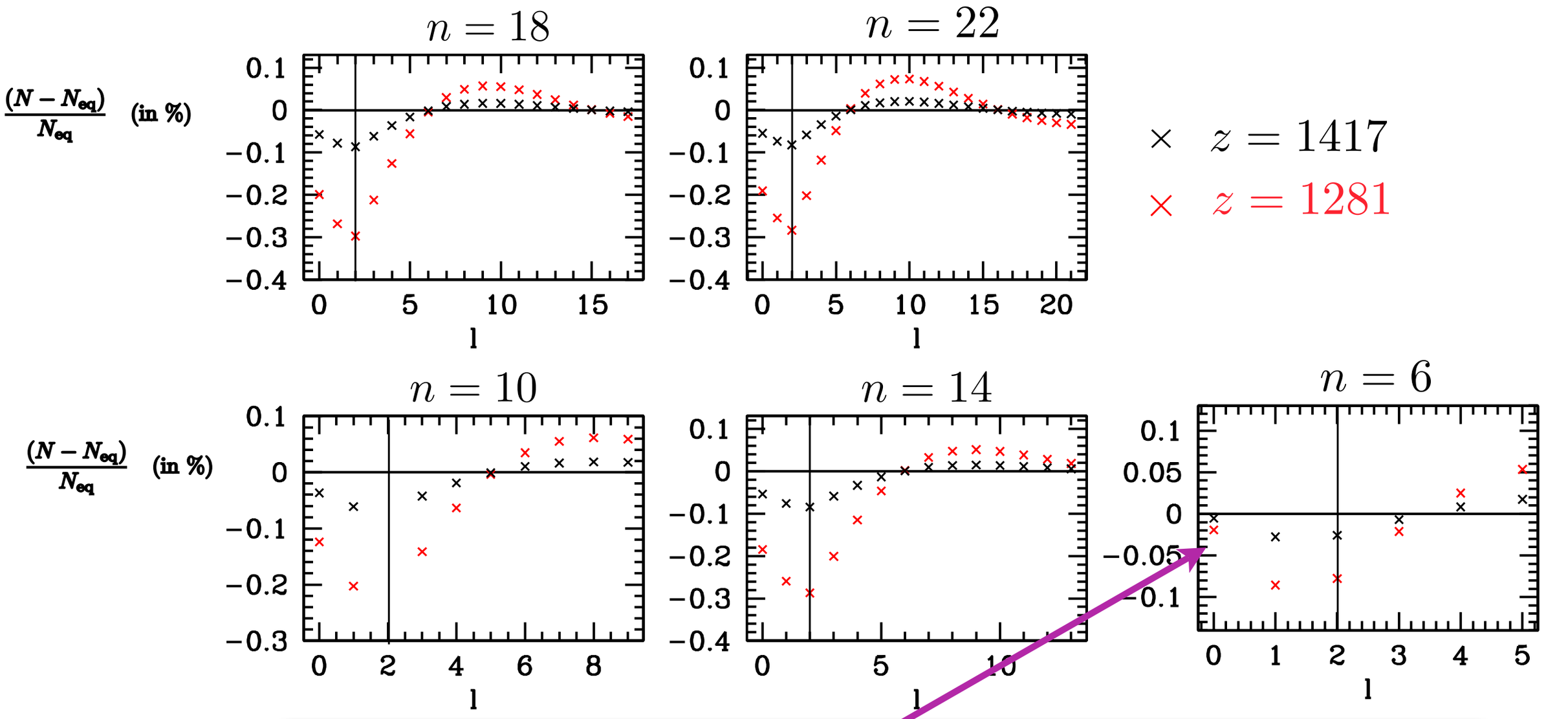
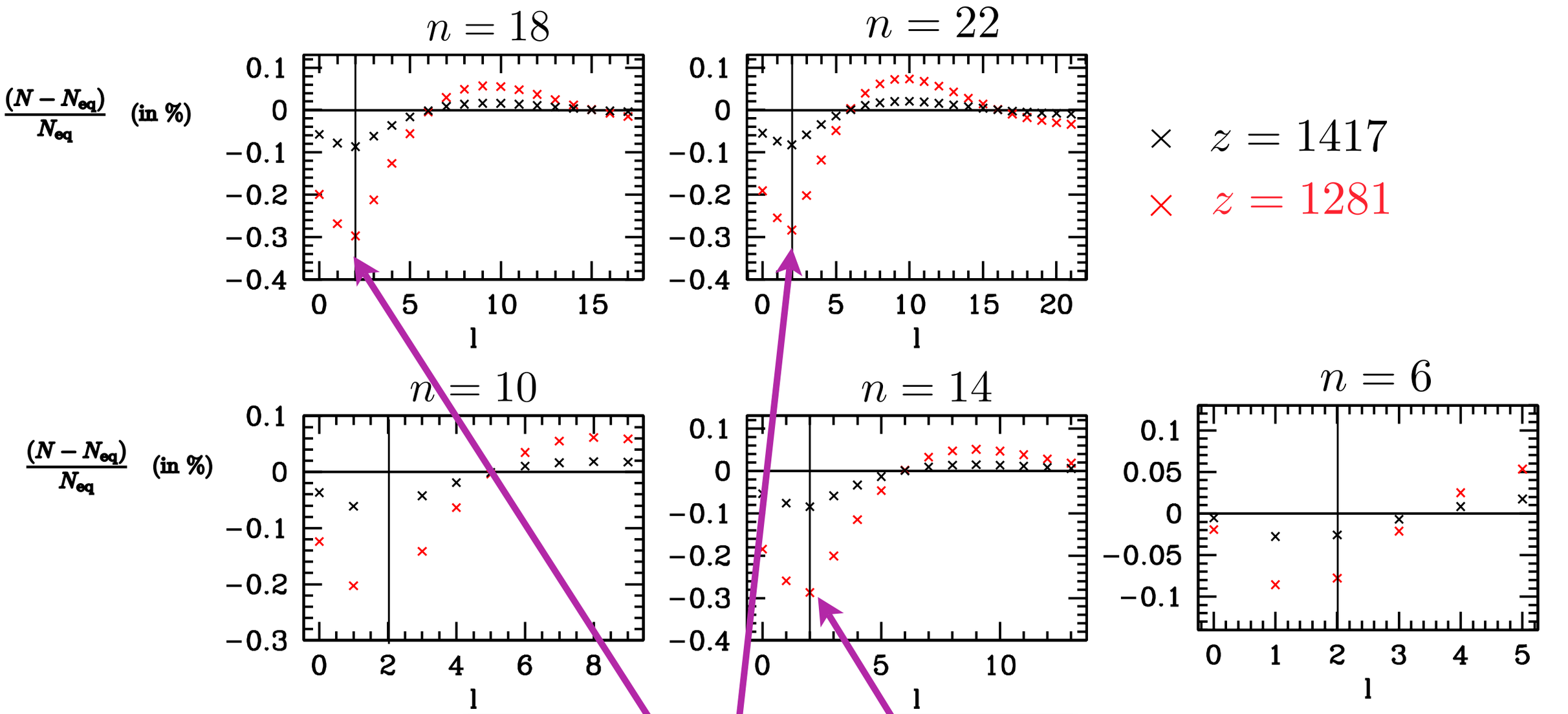


l=0 can't cascade down, so s states are not as under-populated

DEVIATIONS FROM BOLTZMANN EQ: l-substates

RecSparse results $n_{\max} = 30$



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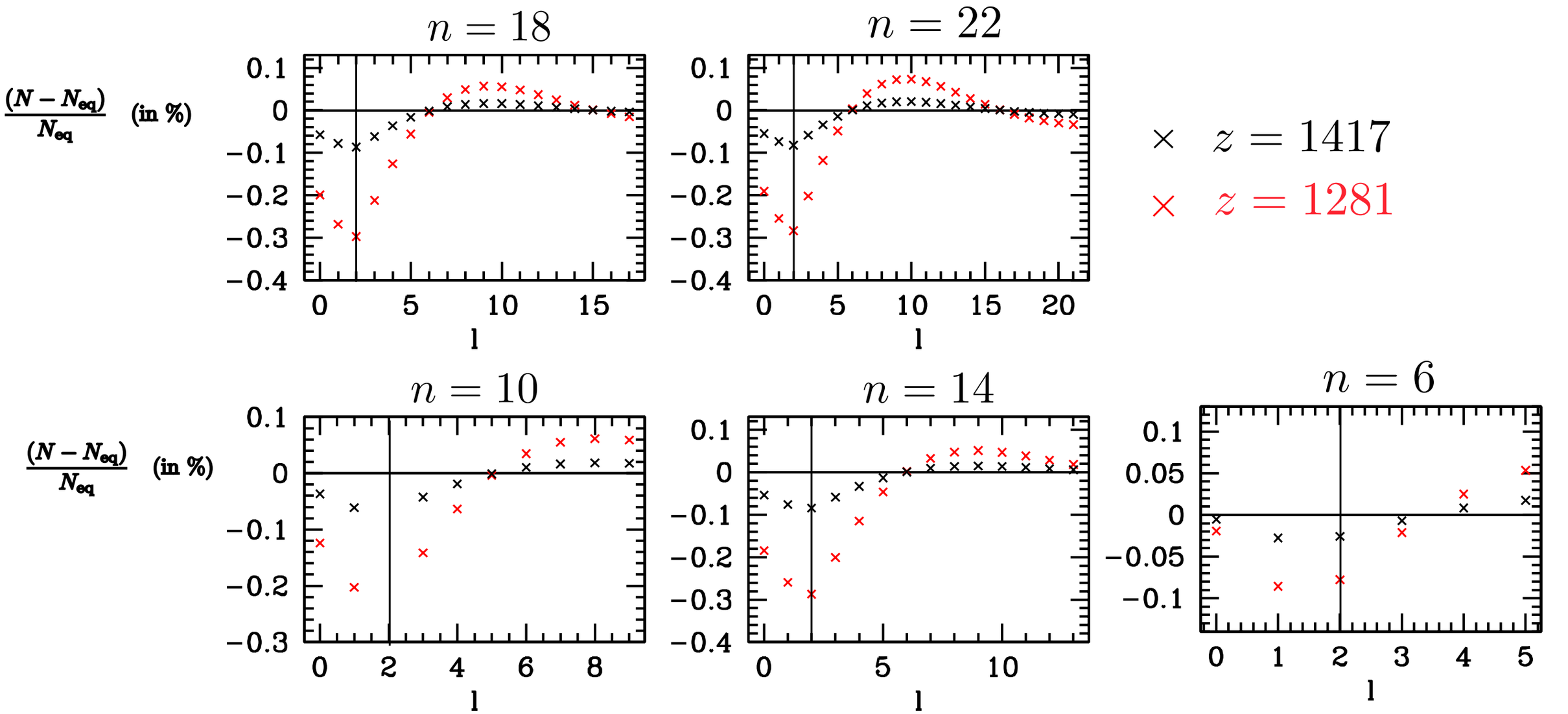
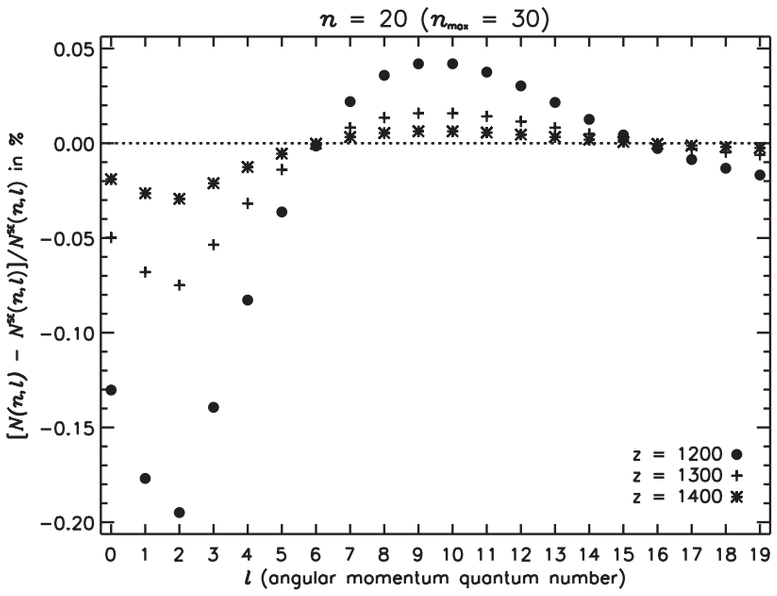
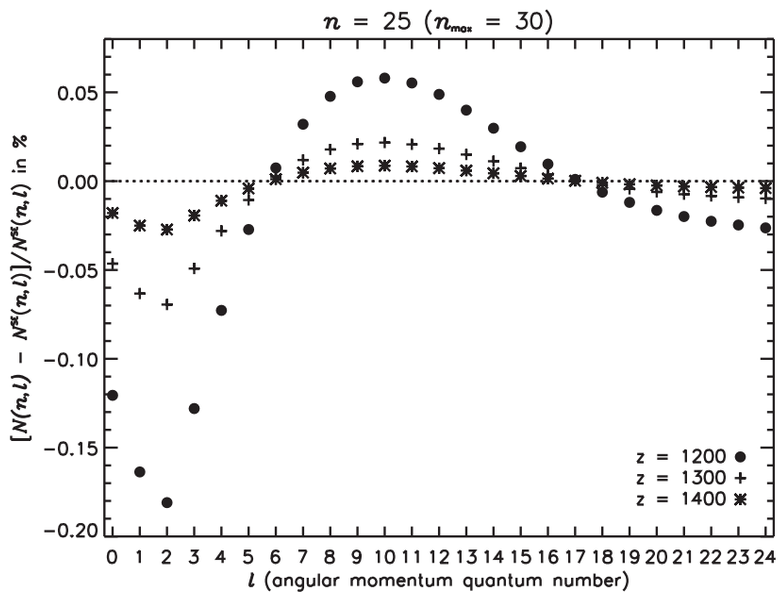
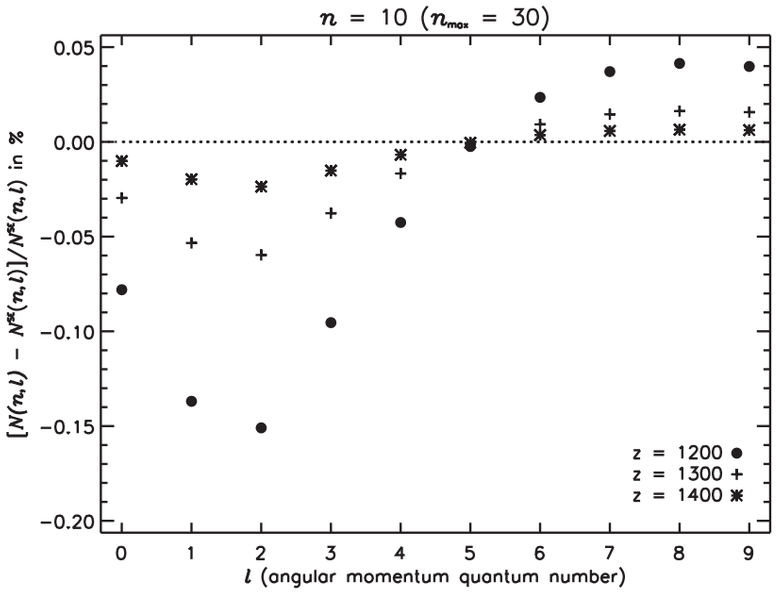


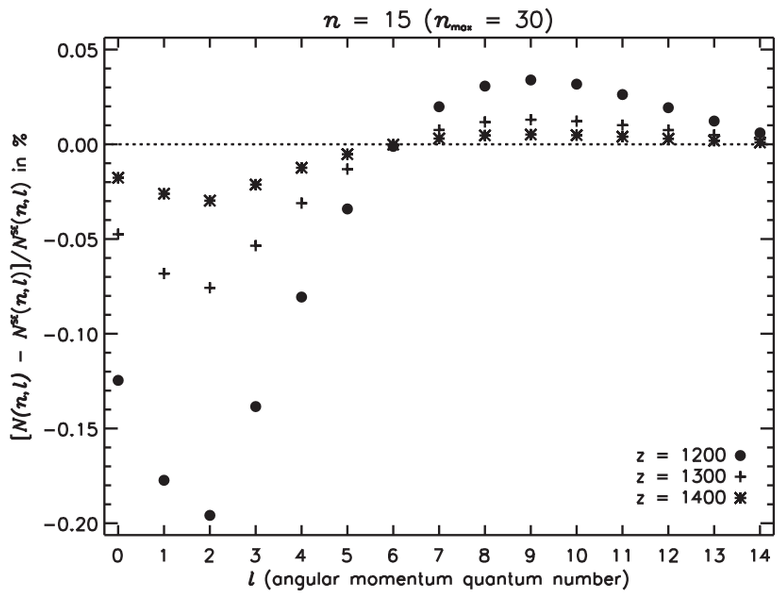
DEVIATIONS FROM BOLTZMANN EQ: l-substates

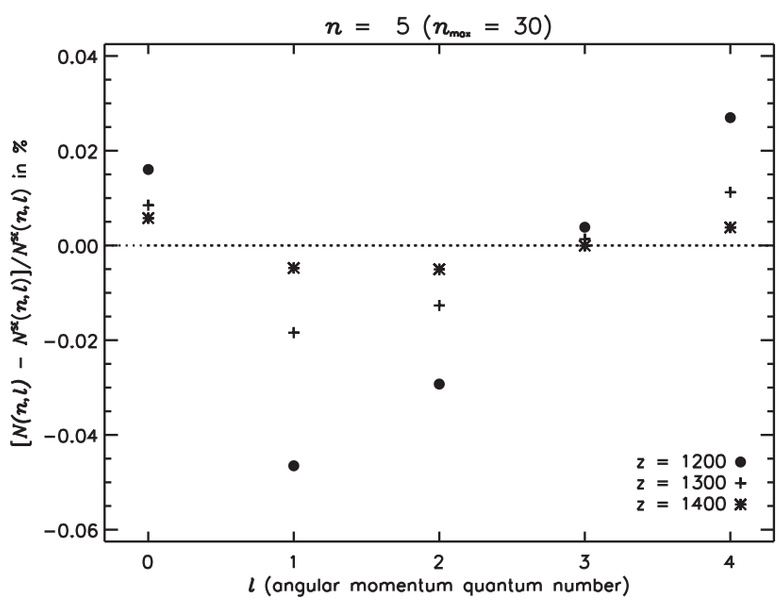
Compare with Rubino-Martin, Chluba, and Sunyaev 2006: Similar Features!





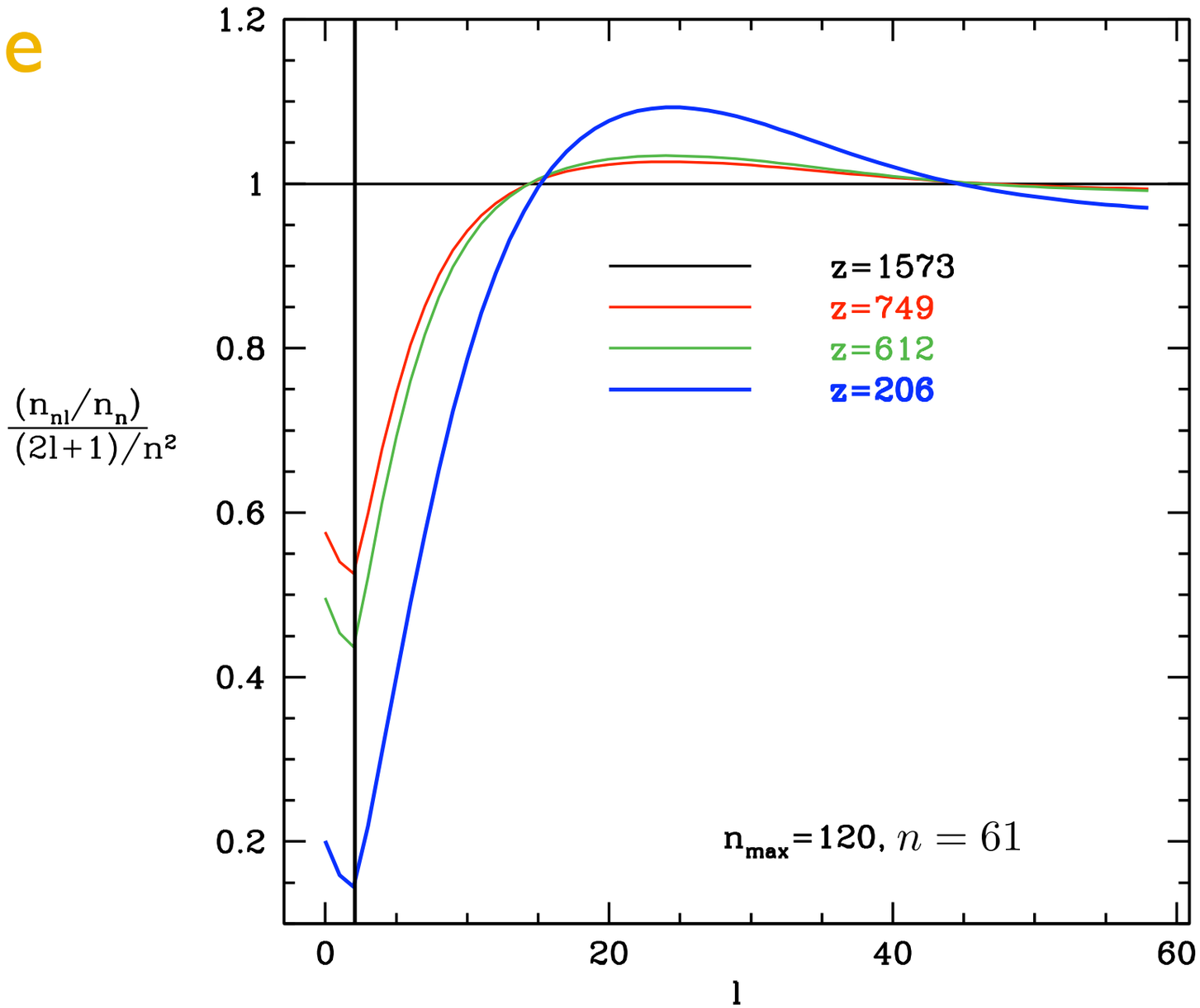






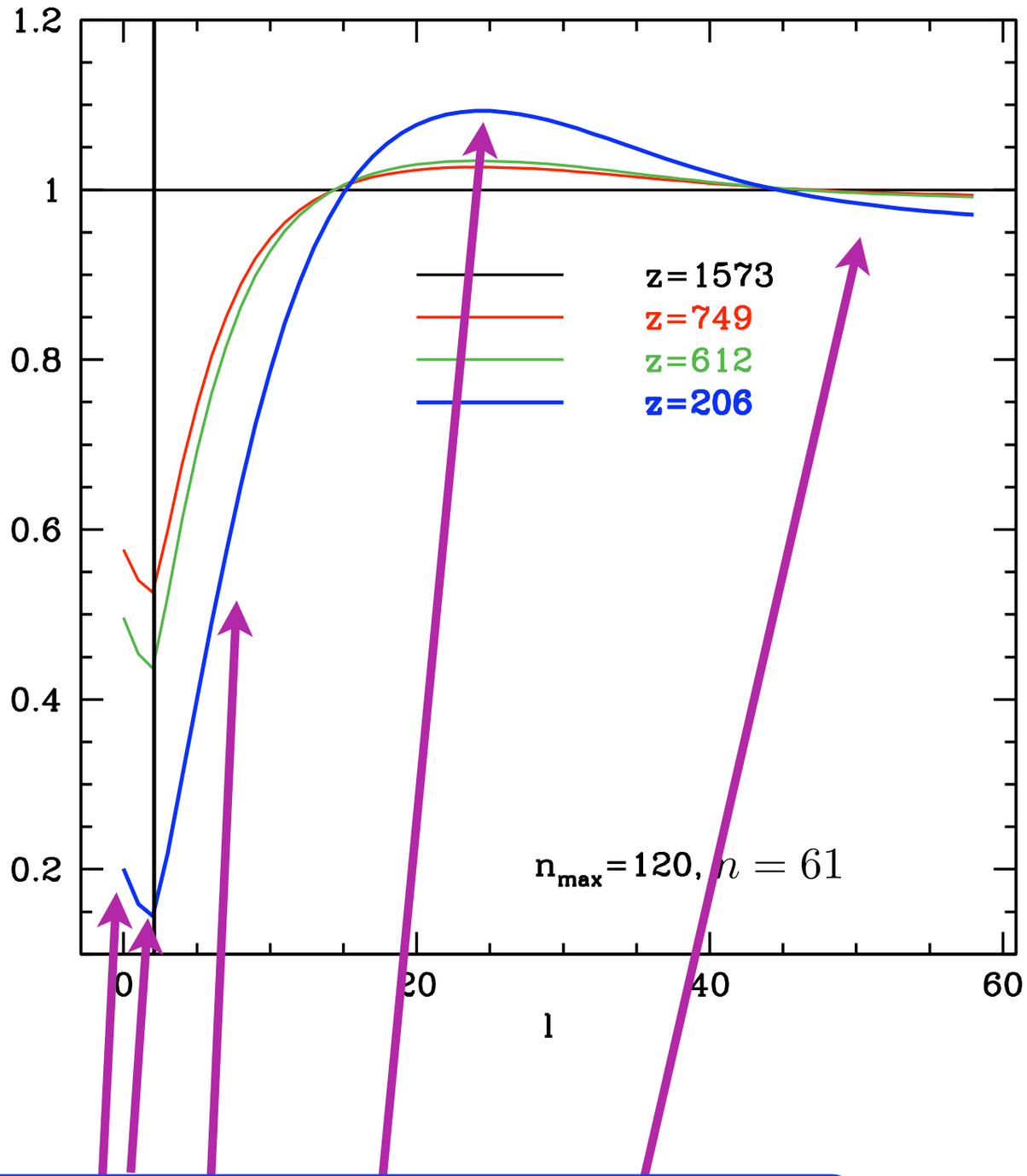
DEVIATIONS FROM BOLTZMANN EQ: l-substates

RecSparse output



DEVIATIONS FROM BOLTZMANN EQ: l-substates

RecSparse output



Patterns persist for high n, $n_{\max}$

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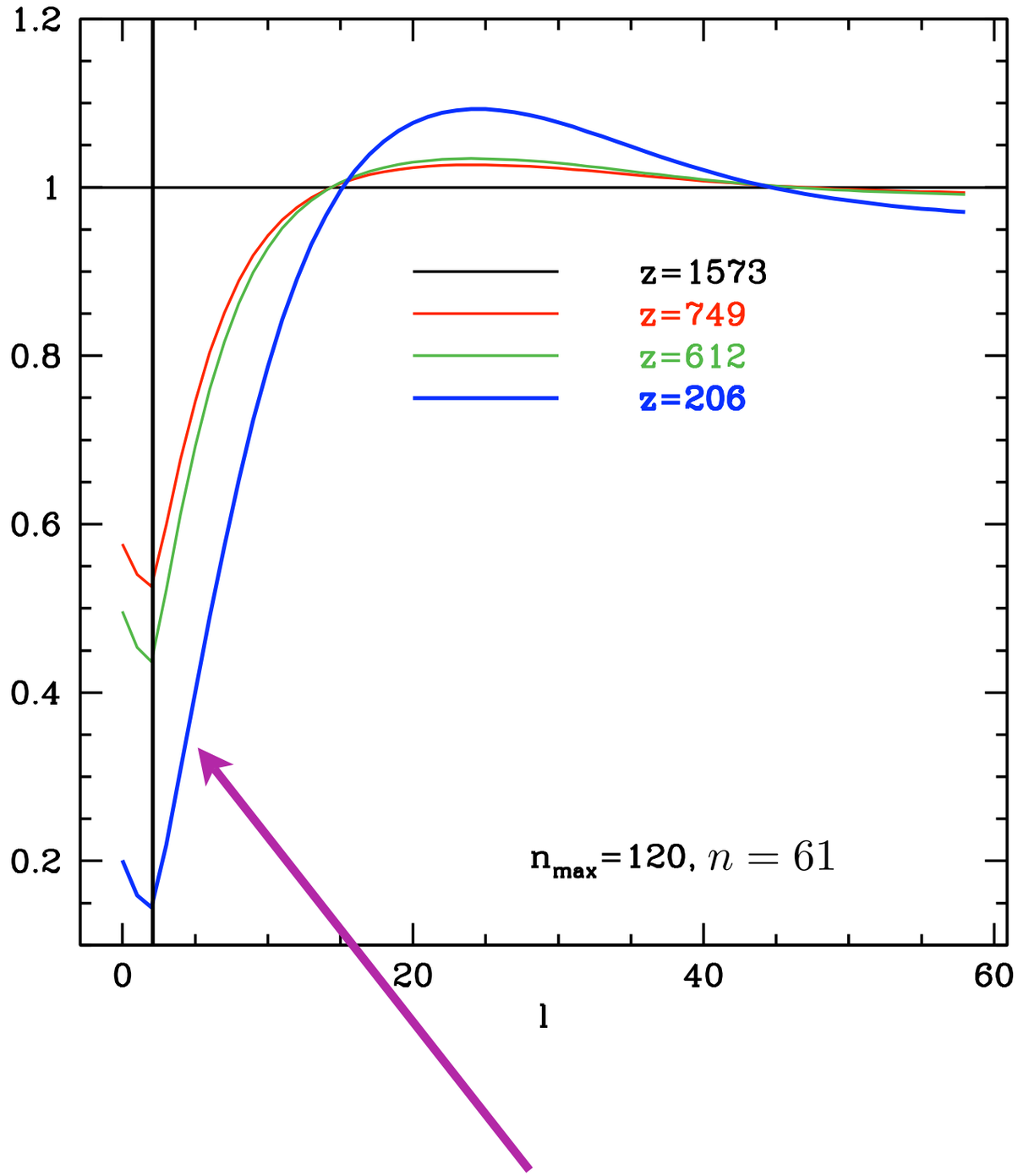


l-substates are highly out of Boltzmann eqb'm at late times

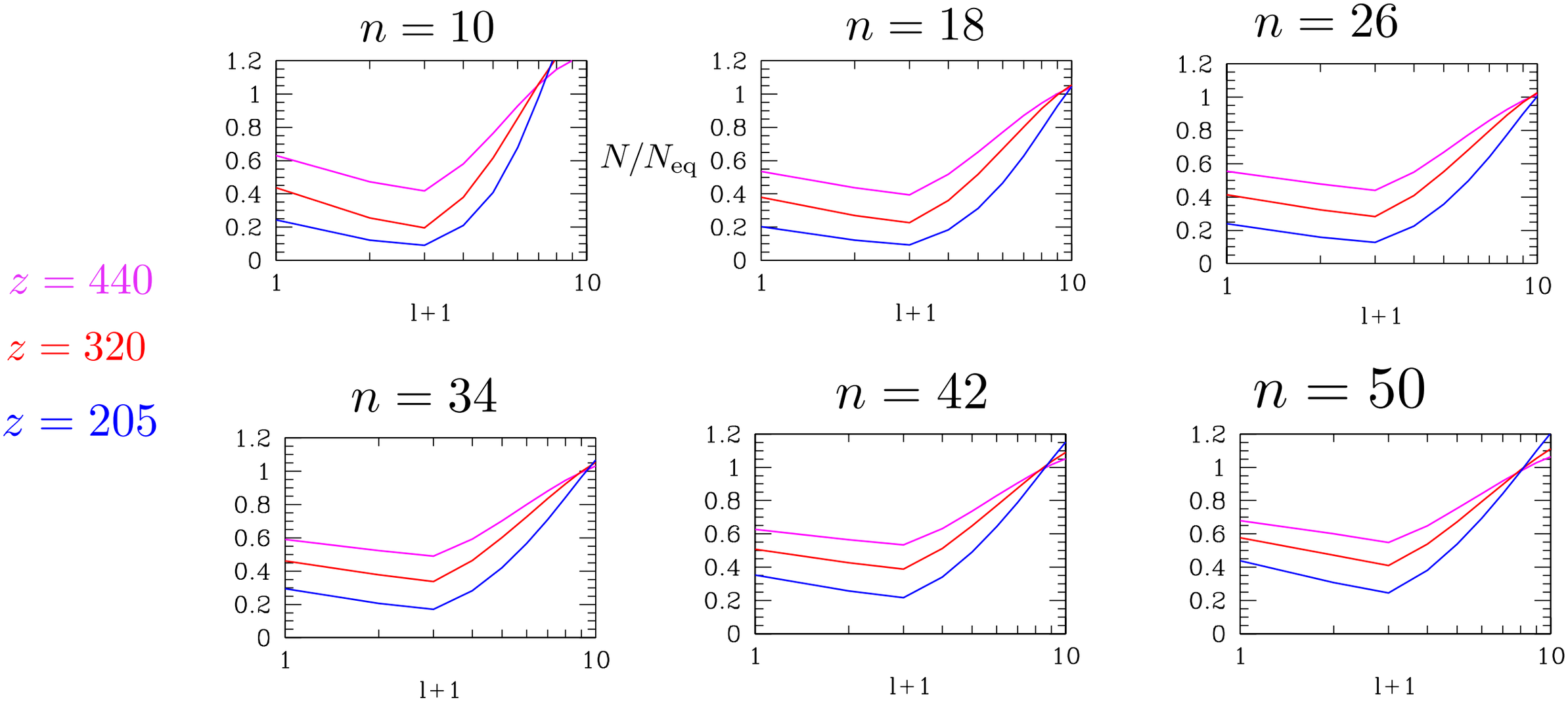
What is the origin of the l=2 dip?

$$A_{\mathrm{nd}\rightarrow\mathrm{2p}} > A_{\mathrm{np}\rightarrow\mathrm{2s}} > A_{\mathrm{ns}\rightarrow\mathrm{2p}}$$

- l=2 depopulates more efficiently than l=1 for higher (n>2) excited states
- We can test if this explains the dip at l=2 by running the code with Balmer transitions from l=2 artificially disabled: the blip should move to l=1

l-substate populations, Balmer lines on

$n_{\rm max} = 50$



$z = 440$

$z = 320$

$z = 205$

l-substate populations, Balmer lines off

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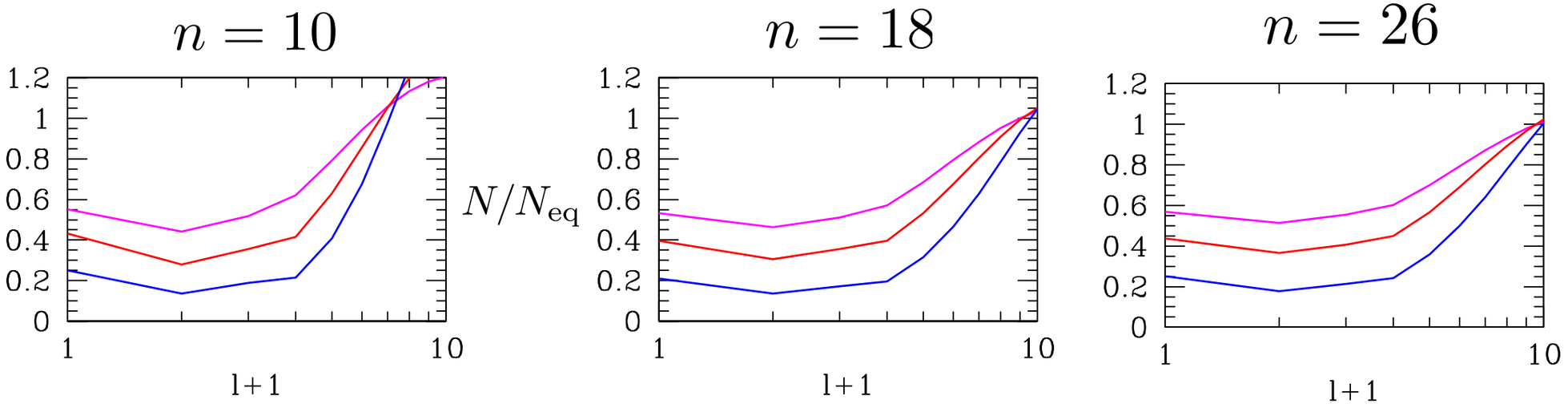

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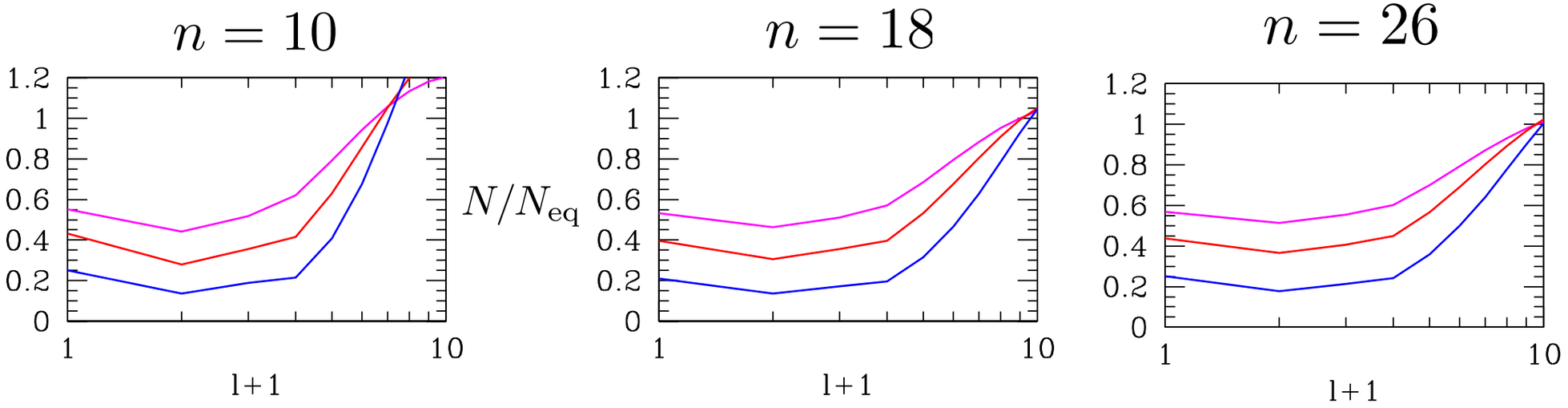

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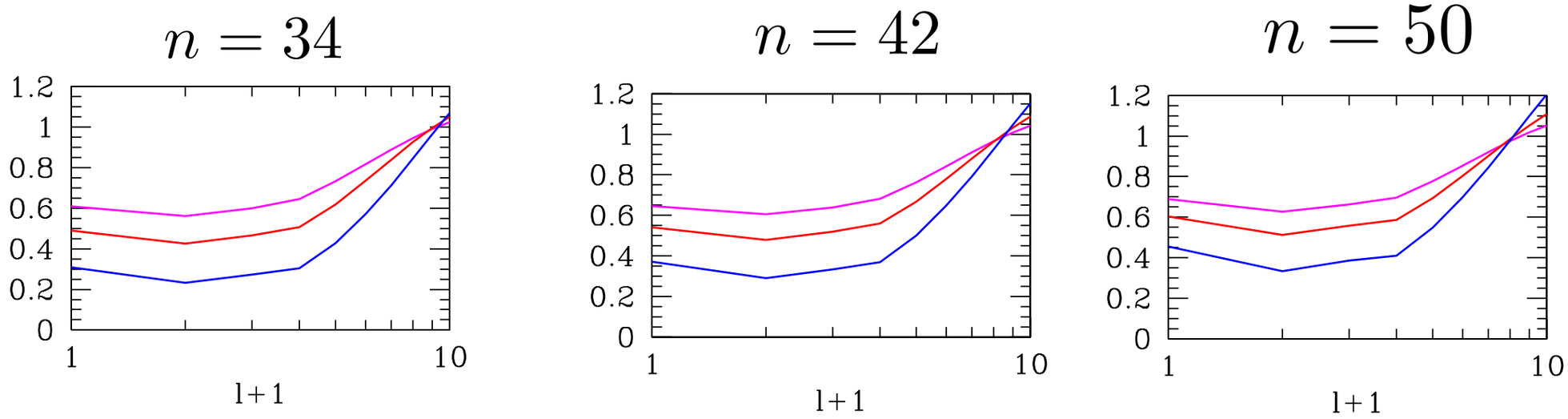

Dip moves as expected when Balmer lines are off!

l-substate populations, Balmer lines off

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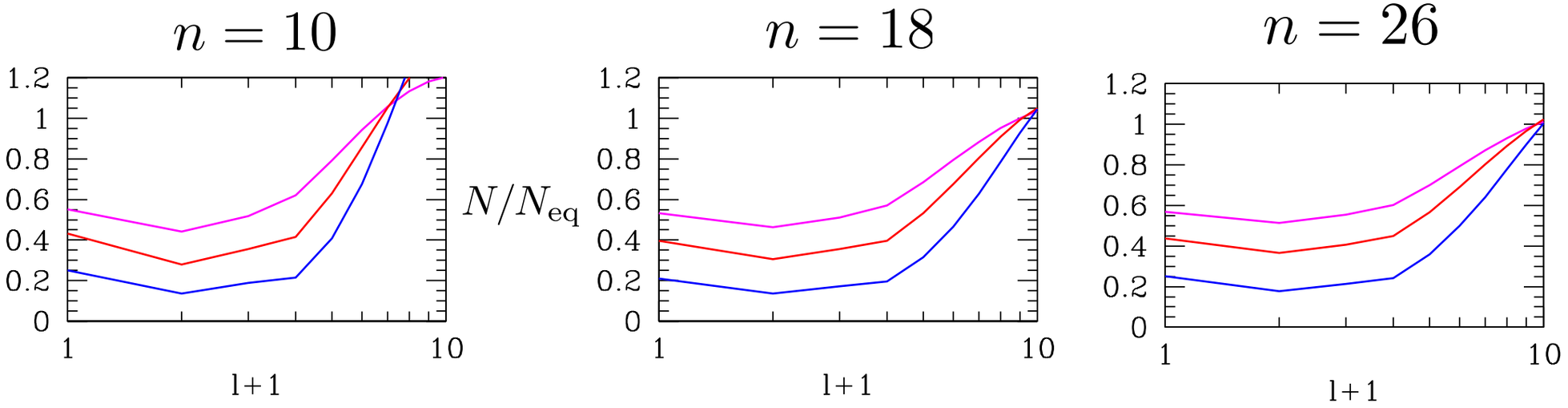

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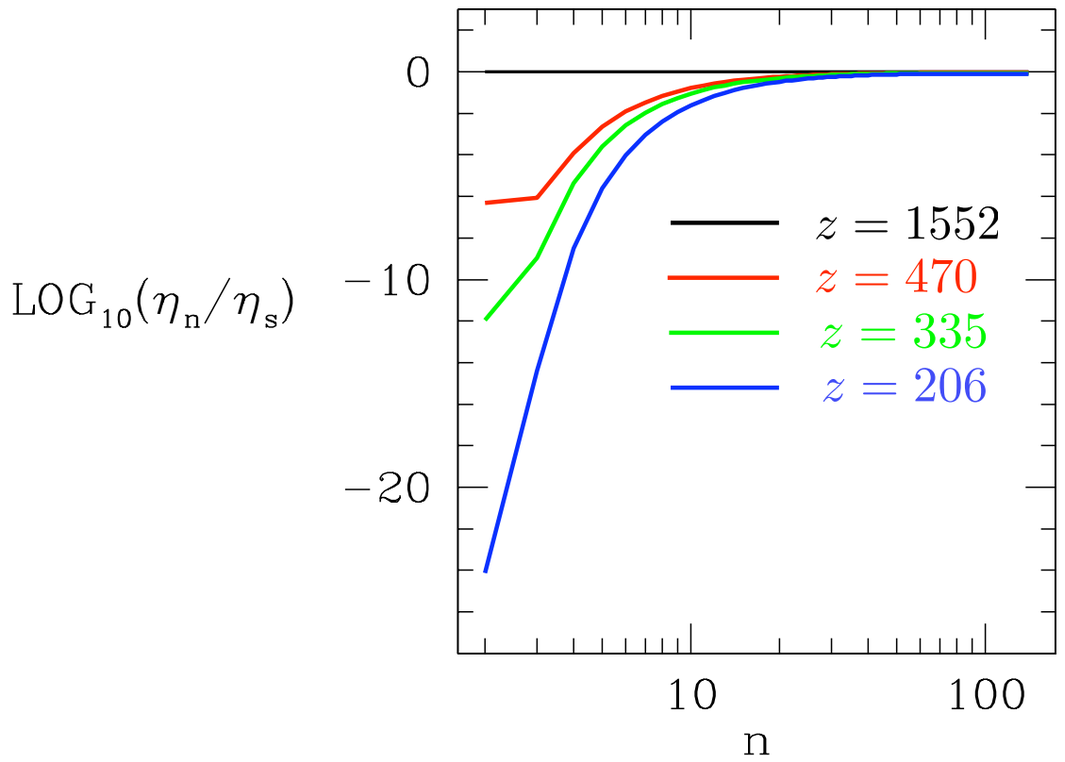
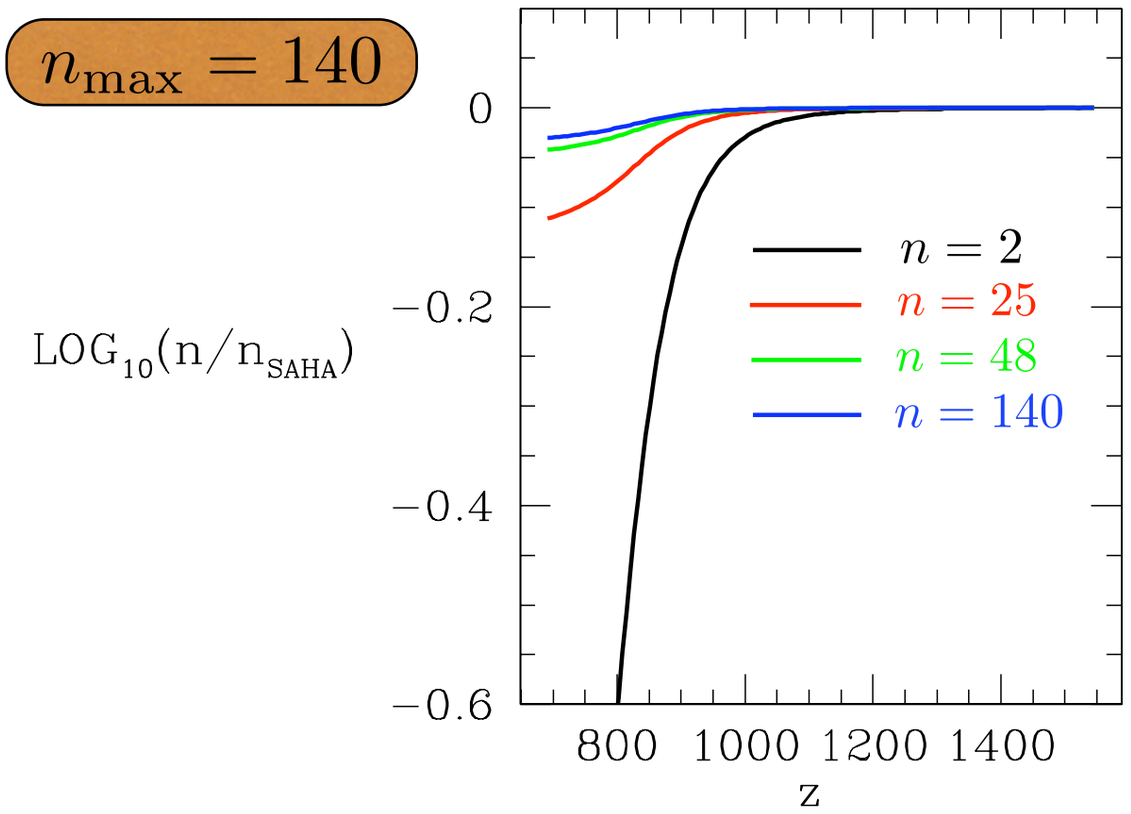
DEVIATIONS FROM SAHA EQUILIBRIUM

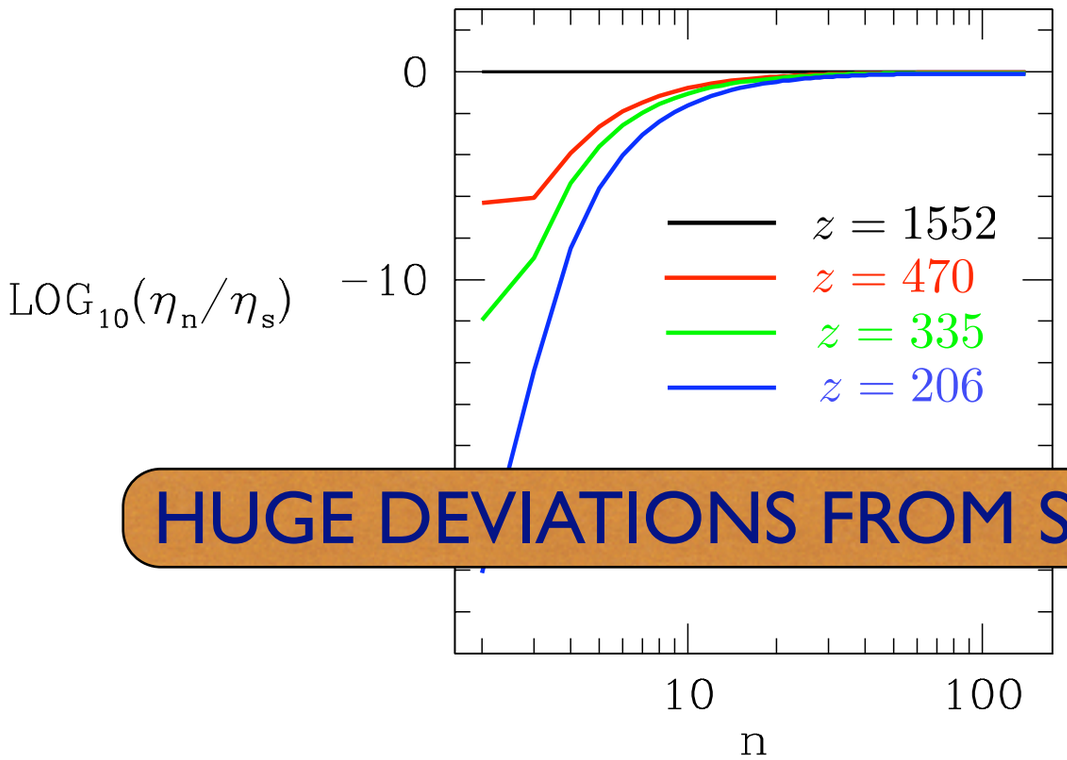


$n_{\max} = 140$



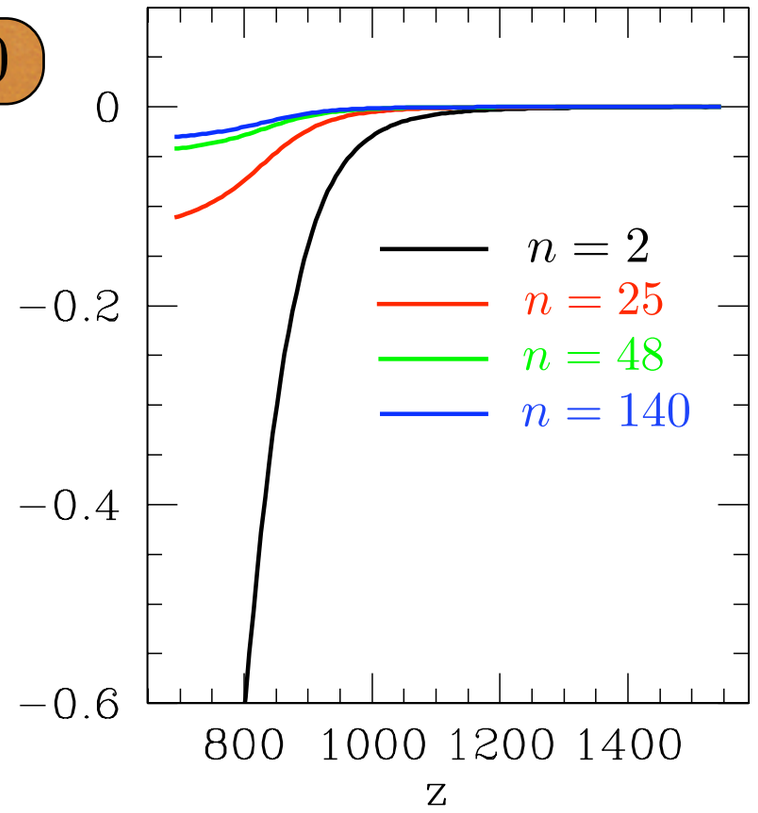
- n=1 suppressed due to freeze-out of x_e
- Remaining levels ‘try’ to remain in Boltzmann eq. with n=2
- Super-Boltz effects and two-γ transitions (n=1 $\longrightarrow$ n=2) yield less suppression for n>1
- Problem gets worse at late times (low z) as rates fall

DEVIATIONS FROM SAHA EQUILIBRIUM



$n_{\max} = 140$

HUGE DEVIATIONS FROM SAHA EQ



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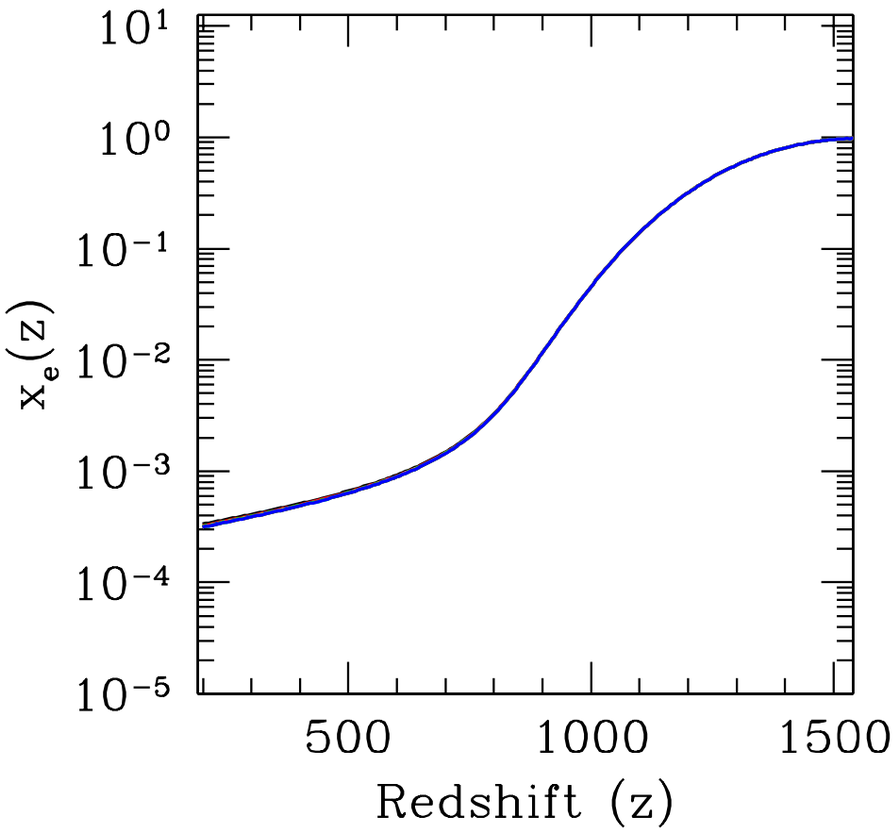
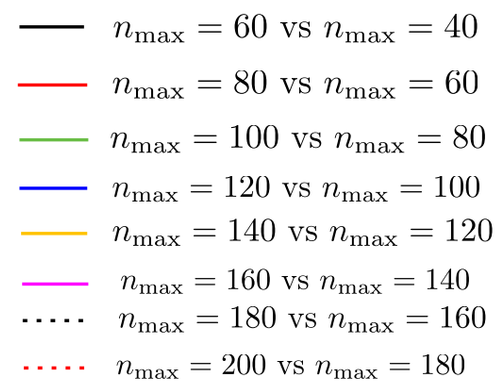
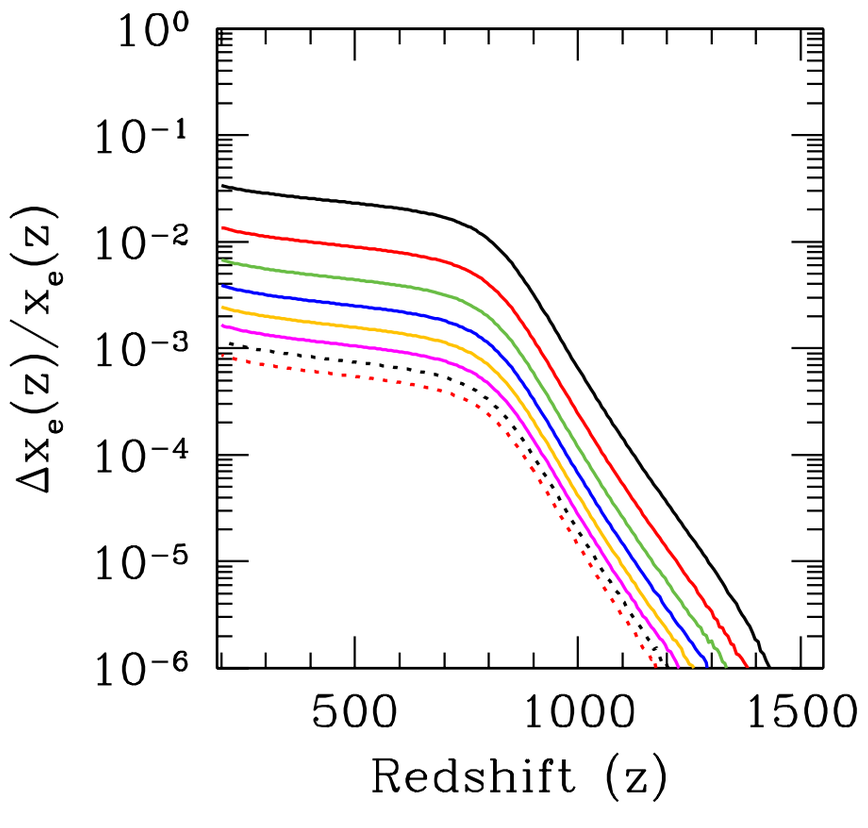
Results: Recombination histories







- $x_e(z)$ falls with increasing $n_{\max} = 10 \rightarrow 200$, as expected.
- Rec Rate>downward BB Rate> Ionization, upward BB rate
- For $n_{max} = 100$, code computes in only 2 hours

QUADRAPOLE TRANSITIONS AND RECOMBINATION

- Electric quadrupole (E2) transitions are suppressed but conceivably not irrelevant at the desired level of accuracy:

$$\frac{A^{\rm quad}_{m,l\pm2\to n,l}}{A^{\rm dipole}_{m,l\pm1\to n,l}} \sim \alpha^2 \approx 5\times10^{-5}$$

- Coupling to ground state will overwhelmingly dominate:

$$\frac{A^{\rm quad}_{n,2\to1,0}}{A^{\rm quad}_{n,2\to m,0}} \propto \frac{\omega^5_{n1}}{\omega^5_{nm}} = \left(\frac{1-\frac{1}{n^2}}{\frac{1}{m^2}-\frac{1}{n^2}}\right)^5 \geq 1024 \text{ if } m\geq 2$$

- Magnetic dipole rates suppressed by several more orders of magnitude

QUADRUPOLE RATES: BASIC FORMALISM

- $$A^{\mathrm{quad}}_{n_a,l_a\to n_b,l_b} = \frac{\alpha}{15}\frac{1}{2l_a+1}\frac{\omega_{ab}^5}{c^4}\left\langle l_a||C^{(2)}||l_b\right\rangle^2\left({}^2R^{n_a l_a}_{n_b l_b}\right)^2$$
- Reduced matrix element evaluated using Wigner 3J symbols:

$$\left\langle l_a||C^{(2)}||l_b\right\rangle = (-1)^{l_a}\sqrt{(2l_a+1)\,(2l_b+1)}\begin{pmatrix} l_a & 2 & l_b \\ 0 & 0 & 0 \end{pmatrix}$$

- Radial matrix element evaluated using operator methods

$${}^2R^{n_a l_a}_{n_b l_b} \equiv \int_0^\infty r^4 R_{n_a l_a}(r) R_{n_b l_b}(r) dr$$

QUADRUPLE TRANSITIONS AND RECOMBINATION

- Lyman lines are optically thick, so $nd \rightarrow 1s$ immediately followed by $1s \rightarrow np$, so this can be treated as an effective $d \rightarrow p$ process with rate $A_{nd \rightarrow 1s} x_{nd}$.
- Preserves sparsity pattern of rate matrix
- Detailed balance yields net rate

$$R^{\rm quad}_{nd \rightarrow np} = A_{nd \rightarrow 1s} \left(x_{nd} - \frac{5}{3} x_{np} \right)$$

- .x$_{3d} > \frac{5}{3} x_{3p}$, so net is $3d \rightarrow 3p$. $3p \rightarrow 2s$ is fast, and $2s \rightarrow 1s$ dominates recombination rate at early times, so this accelerates recombination.
- For $n > 3, x_{nd} < \frac{5}{3} x_{np}$, so net is $np \rightarrow nd$.$nd \rightarrow 2p$ is fast, but $2p \rightarrow 1s$ is a slow recombination channel while optically thick. As it overtakes $2s \rightarrow 1s$, higher quadrupoles also accelerate recombination.

Quadrapole rates: Operator algebra

- Radial Schrödinger equation can be factored to yield:

$$ {}^{-}\Omega_{nl} = \frac{1}{l A_{nl}} \left[1 - l \left(\frac{d}{dr} + \frac{l+1}{r} \right) \right] \qquad {}^{+}\Omega_{nl} = \frac{1}{l A_{nl}} \left[1 + l \left(\frac{d}{dr} - \frac{l-1}{r} \right) \right] $$

$$ {}^{-}\Omega_{nl} R_{nl}(r) = R_{n\ l-1}(r) $$

$$ {}^{+}\Omega_{n\ l-1} R_{nl}(r) = R_{nl}(r) $$

$$ A_{nl} = \frac{\sqrt{n^2 - l^2}}{nl} $$

- This algebra can be applied to radial matrix elements:

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- This algebra can be applied to radial matrix elements:

$$ {}^{2}R^{n\ l-1}_{n'\ l-1} = \frac{1}{A_{nl}} \left\{ A_{n'l}\, {}^{2}R^{nl}_{n'l} + 2\, {}^{(1)}R^{nl}_{n'\ l-1} \right\} \qquad {}^{(2)}R^{n\ n'-1}_{n'\ n'-1} = \frac{2nn'}{\sqrt{n^2 - n'^2}}\, {}^{(1)}R^{nn'}_{n\ n'-1} $$

Diagonal!

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- This algebra can be applied to radial matrix elements:

$$ l(2l+3) A_{n'l}\, {}^{(2)}R^{n\ l+1}_{n'\ l-1} = (2l+1)(l+2) A_{n\ l+2}\, {}^{(2)}R^{n\ l+2}_{n'l} + 2(l+1) A_{n'\ l+1}\, {}^{(2)}R^{n\ l+1}_{n'\ l+1} + 2(2l+1)(3l+5)\, {}^{(1)}R^{n\ l+1}_{n'l} \quad (1 \le l \le n' - 1) $$

$$ {}^{(2)}R^{n\ n'-1}_{n'\ n'+1} = 0 $$

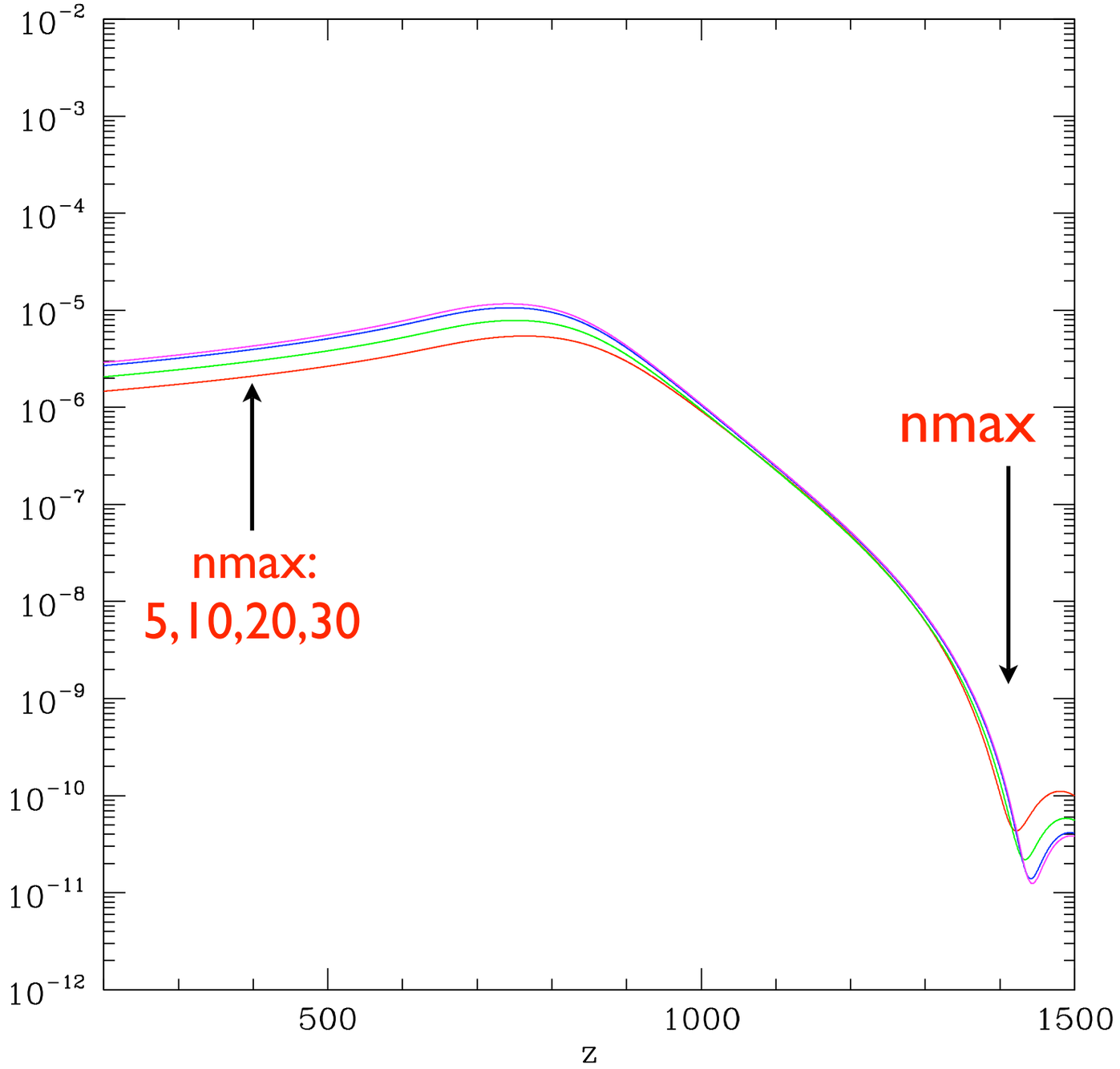
$$ {}^{(2)}R^{n\ n'+1}_{n'\ n'-1} = (-1)^{n-n'} 2^{2n'+4} \left[\frac{(n+n'+1)!}{(n-n'-2)!(2n'-1)!} \right]^{1/2} n' \left(nn' \right)^{n'+3} \frac{\left(n-n' \right)^{n-n'-3}}{\left(n+n' \right)^{n+n'+3}} $$

Off-diagonal!

QUADRUPOLE RATES: VERIFICATION

- Rates were checked using WKB expressions like dipole rates
- Compared to published numerical rates of Jitrik and Bunge: 4-5 digits of agreement (Dirac vs. non-rel wf), but this would be a correction to a small correction

RESULTS: QUADRUPOLE RATES AND RECOMBINATION



Who Cares?

I. Smearing and moving the surface of last scattering (SLSS)

- Photons kin. decouple when Thompson scattering freezes out

$$\gamma + e^- \Leftrightarrow \gamma + e^-$$

$$\Gamma = n_e \sigma_{\mathrm{T}} c = 2.2 \times 10^{-19}\ \mathrm{s}^{-1} \frac{x_e \Omega_b h^2}{a^3} =$$

$$H = H_0 \Omega_m^{1/2} a^{-3/2} \left[1 + \frac{a_{\mathrm{eq}}}{a}\right]^{1/2}$$

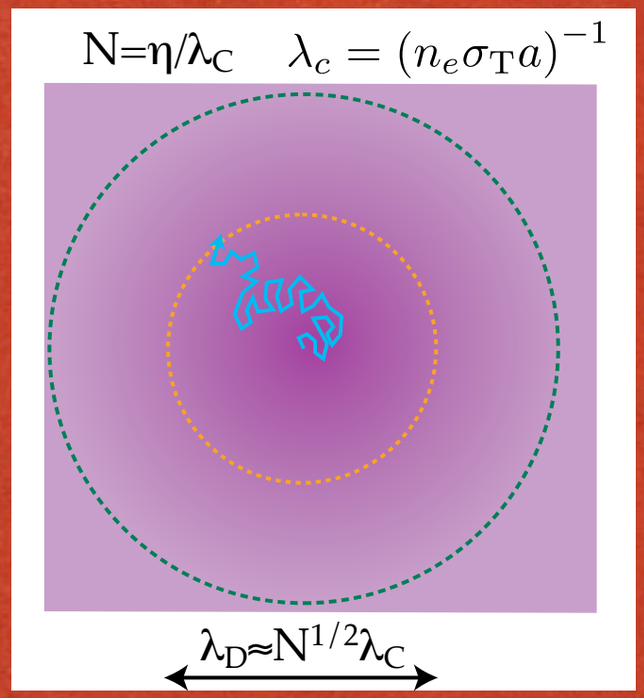
- $z_{\mathrm{dec}} \simeq 1100$:Decoupling occurs during recombination

$$C_l \rightarrow C_l e^{-2\tau} \quad \text{if } l > \frac{\eta_0}{\eta_{\mathrm{rec}}}.$$

$$\tau = \int_0^{\eta_{\mathrm{dec}}} d\eta n_e \left[\eta\right] \sigma_{\mathrm{T}} a\left(\eta\right)$$

Who Cares?
II. The Silk Damping Tail

- From Wayne Hu's website



$$l_{\rm damp} \sim 1000$$

- Inhomogeneities are damped for $\lambda < \lambda_D$

$$k_D^{-2}(\eta) \simeq \int_0^\eta \frac{d\eta'}{6(1+R)n_e[\eta']\sigma_T a[\eta']}\left[\frac{R^2}{1+R}+\frac{8}{9}\right]$$

$$R = \frac{3\rho_b^0}{4\rho^\gamma}$$

$$|\Theta_l(\eta_0)| \simeq \int_0^{\eta_0} d\eta\, \dot\tau e^{-\tau(\eta)} e^{ik\int d\eta c_s} e^{-k^2/k_D^2(\eta)} \tilde\delta(k) j_l(k(\eta-\eta_0))dk$$

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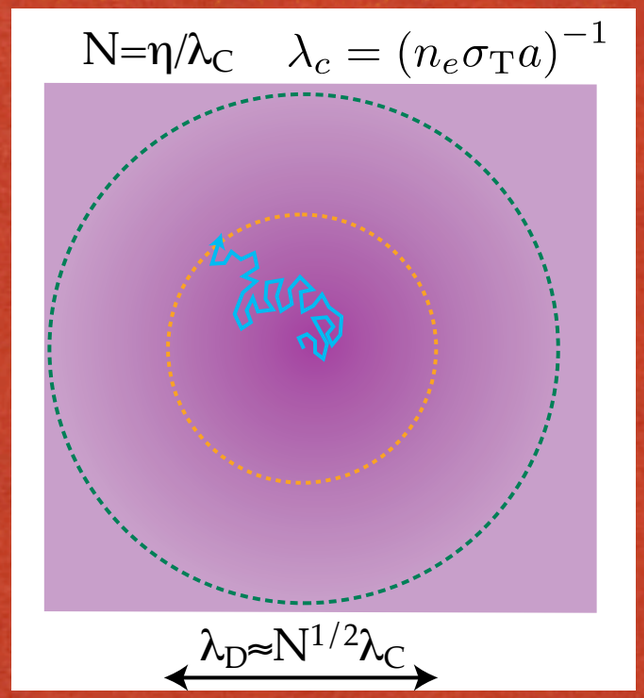

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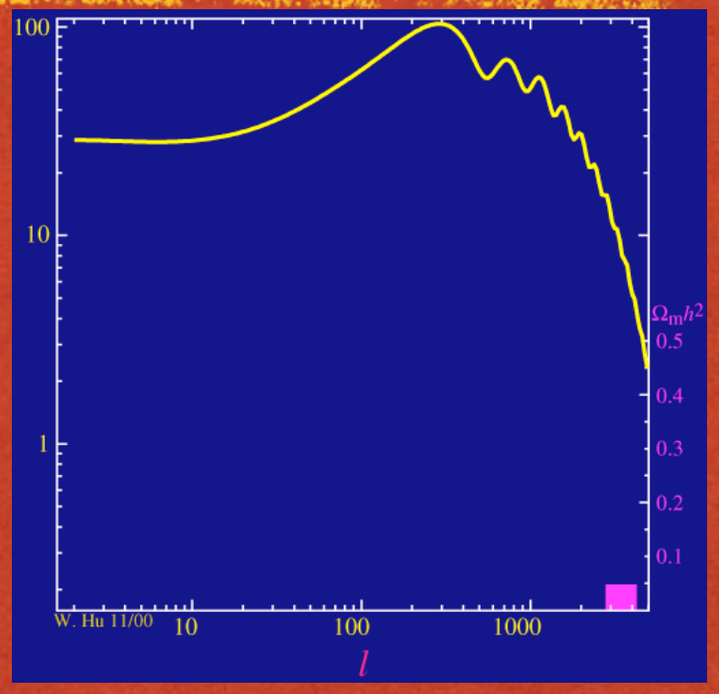

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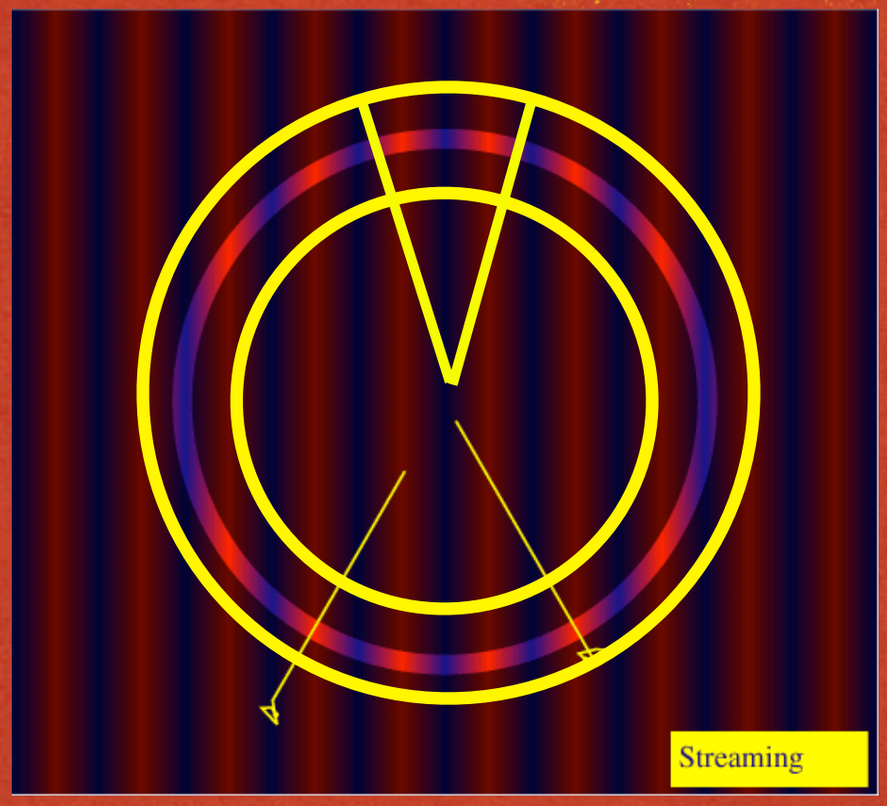
Who Cares?
III. Finite thickness of the SLSS



- Additional damping of form

$$|\Theta_l(\eta_0,k)| \rightarrow |\Theta_l(\eta_0,k)|\, e^{-\sigma^2 \eta_{\rm rec}^2 k^2}$$

Who Cares?
IV. CMB Polarization

- Need to scatter quadrapole to polarize CMB

$$\Theta_l^P(k) = \int d\eta \dot{\tau} e^{-\tau(\eta)} \Theta_{T,2}(k,\eta) \frac{l^2}{(k\eta)^2} j_l(k\eta)$$

- Need time to develop a quadrapole

$$\Theta_l(k\eta) \sim \frac{k\eta}{2\tau} \Theta_l(k\eta) \ll \Theta_l(\eta) \text{ if } l \geq 2, \text{ in tight coupling regime}$$

Who Cares?
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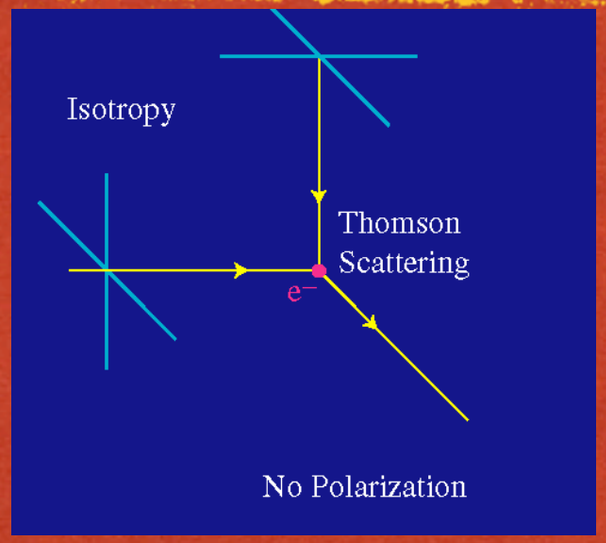

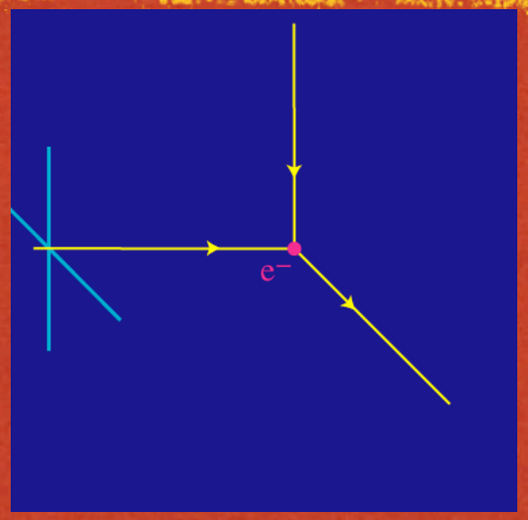

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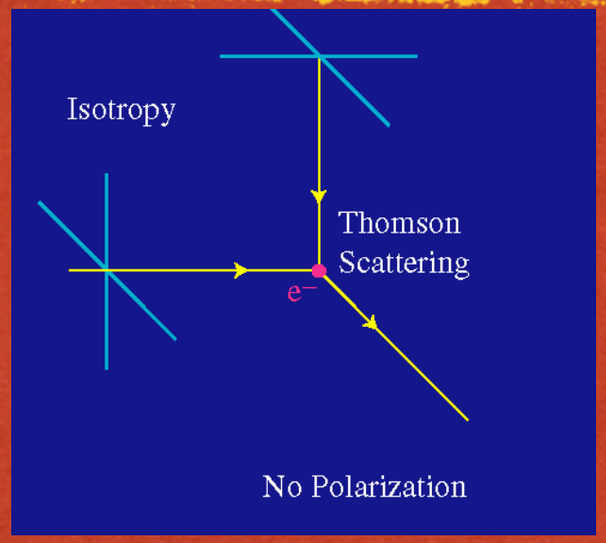

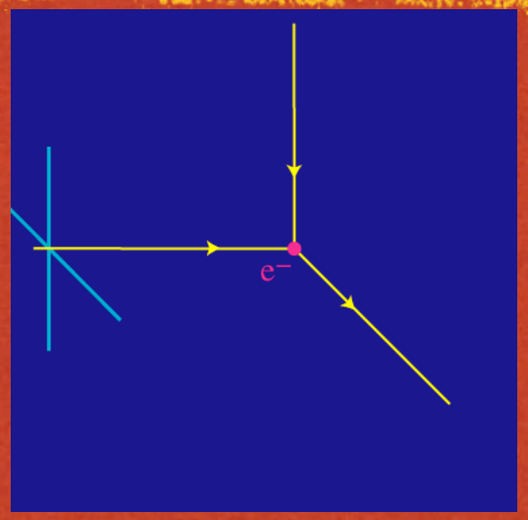

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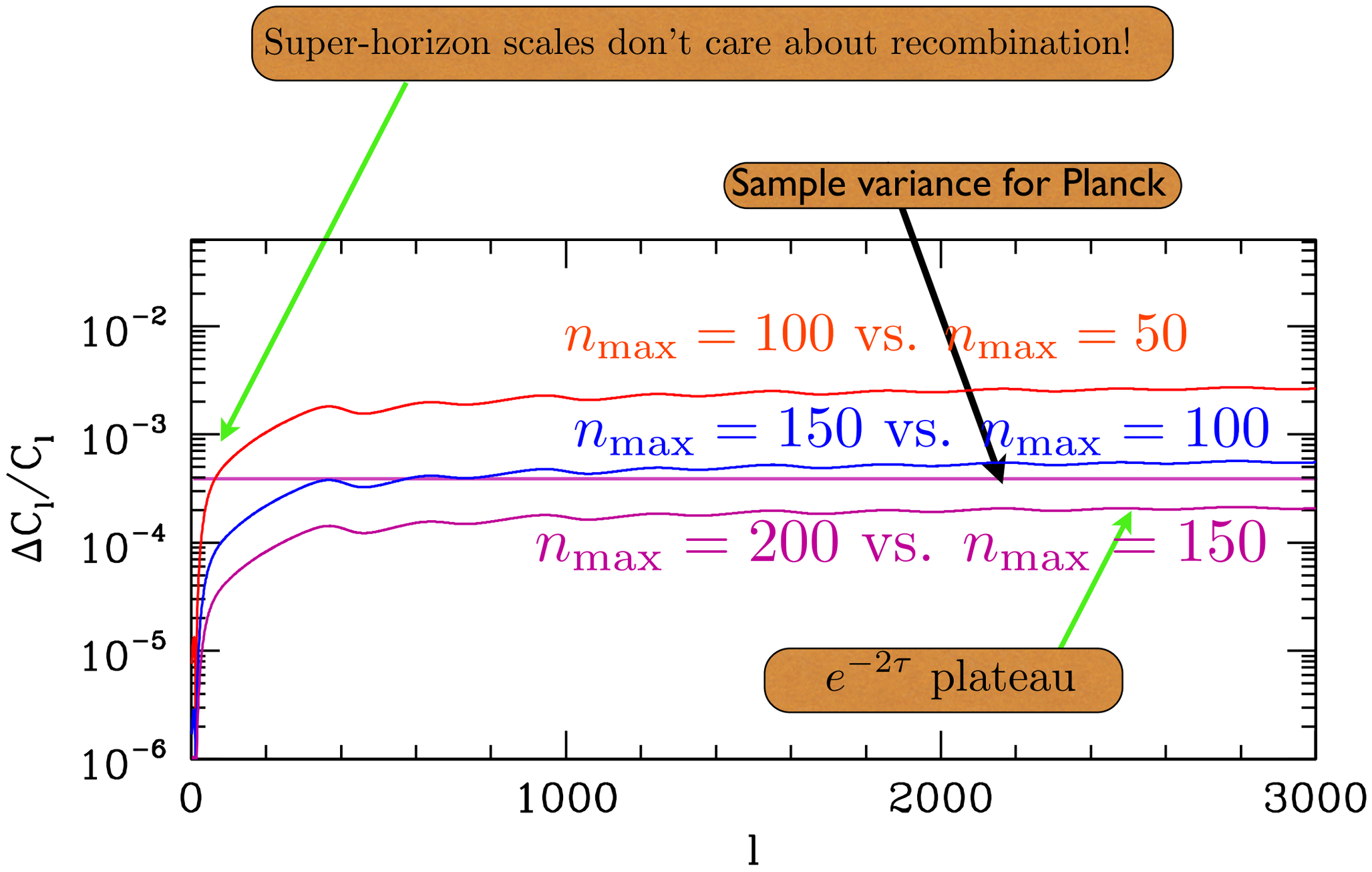
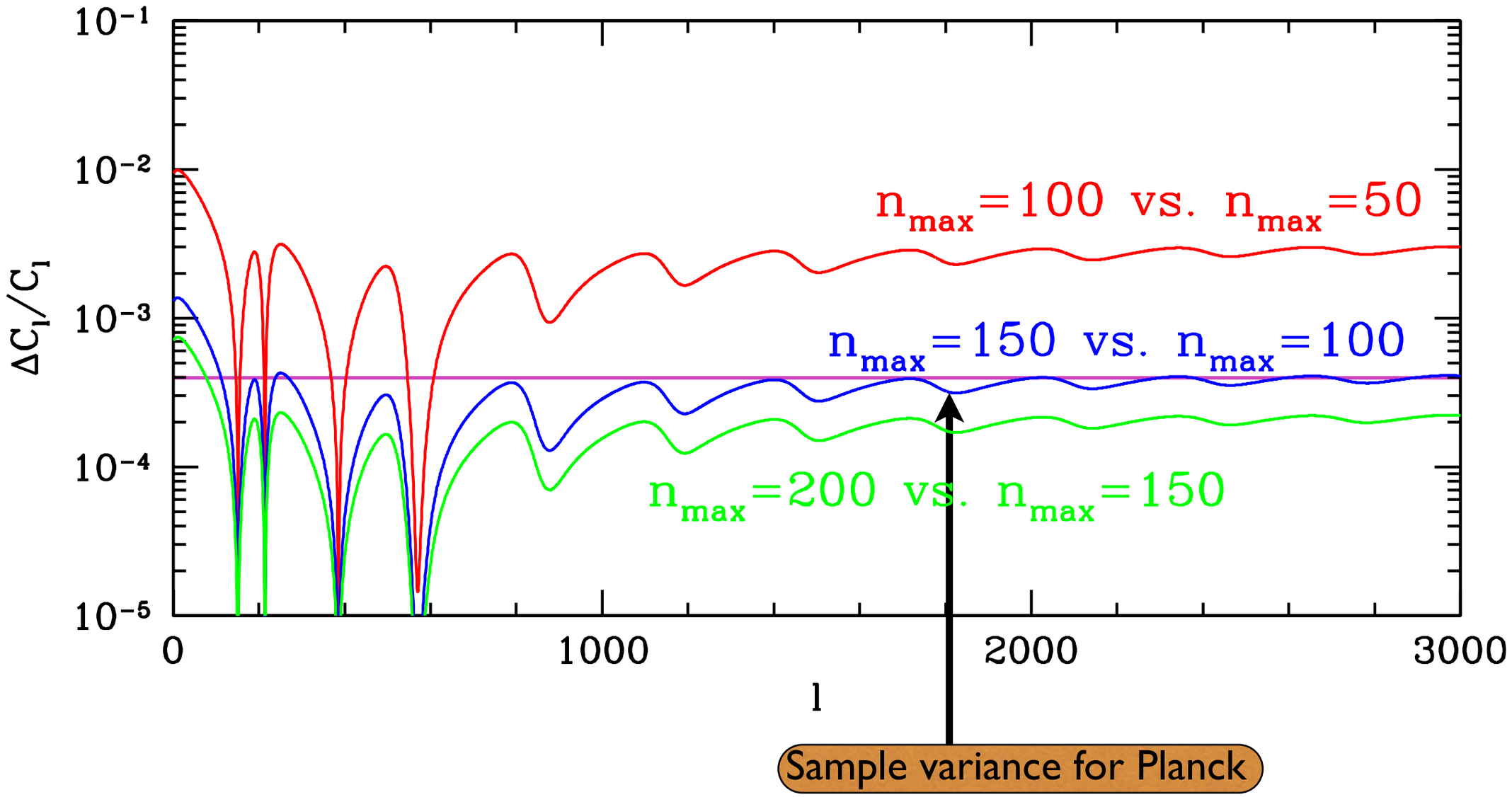
Temperature $C_l s$



EE Polarization $C_l s$

Lower τ after LSS, wider LSS $\rightarrow$ more polarization

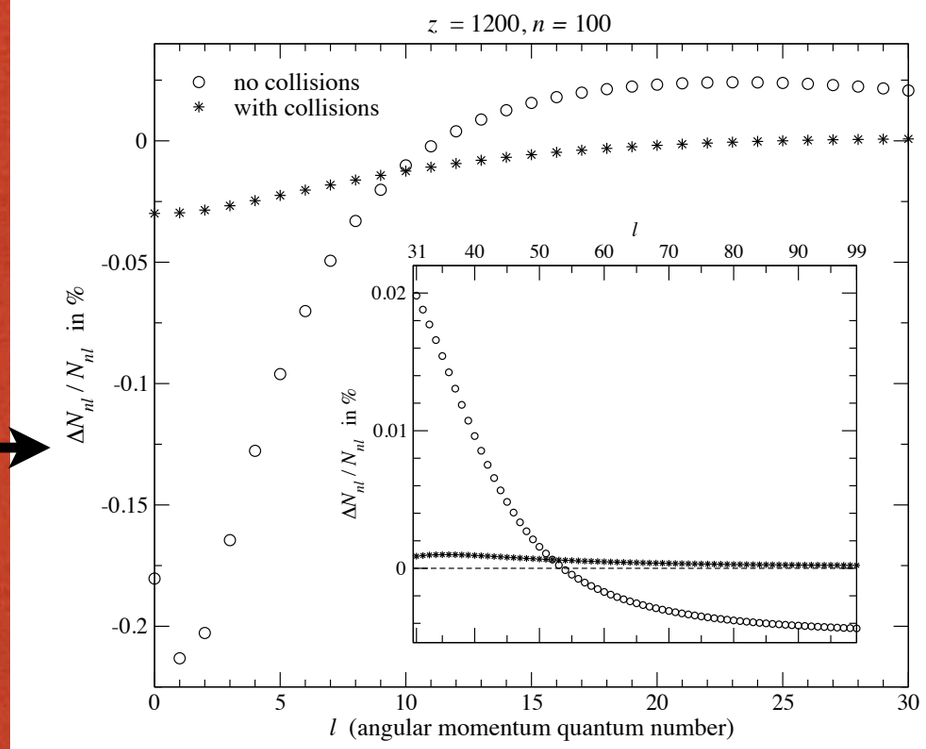


Atomic Collisions

- For fixed n, l-changing collisions bring different-l substates closer to statistical equilibrium (SE)

- Being closer to SE speeds up rec. by mitigating high-l bottleneck (Chluba, Rubino Martin, Sunyaev 2006)

$n_{\max} = 100$



- Theoretical collision rates unknown to factors of 2!
  - $b < a_0 n^2 \rightarrow$ multi-body QM!
  - $t_{\rm pass} < t_{\rm orbit} \rightarrow$ Impulse approximation breaks down!

- Next we'll include them to see if we need to model rates better

Wrapping up

- Start using a more efficient integration
- Incorporation of Yacine's line-overlap formalism in place of Sobolev approximation
- Collisions
- Effective source term for omitted higher levels- near Saha eq., should be tractable
- Full incorporation into CMBFAST/CAMB and analysis of errors/degeneracies with cosmo. parameters